# Introduction to Random Processes
# UDRC Summer School, 27th June 2016

*Accompanying Notes*

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# Copyright Statement



These lecture notes consist of entirely original work, where all material has been written and typeset by the author. No figures or substantial pieces of text has been reproduced verbatim from other texts.

However, there is some material that has been based on work in a number of previous textbooks, and therefore some sections and paragraphs have strong similarities in structure and wording. These texts have been referenced and include, amongst a number of others, in order of contributions:

- Manolakis D. G., V. K. Ingle, and S. M. Kogon, *Statistical and Adaptive Signal Processing: Spectral Estimation, Signal Modeling, Adaptive Filtering and Array Processing*, McGraw Hill, Inc., 2000.

  **IDENTIFIERS** – *Paperback*, ISBN10: 0070400512, ISBN13: 9780070400511

- Therrien C. W., *Discrete Random Signals and Statistical Signal Processing*, Prentice-Hall, Inc., 1992.

  **IDENTIFIERS** – *Paperback*, ISBN10: 0130225452, ISBN13: 9780130225450
  *Hardback*, ISBN10: 0138521123, ISBN13: 9780138521127

- Kay S. M., *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall, Inc., 1993.

  **IDENTIFIERS** – *Hardback*, ISBN10: 0133457117, ISBN13: 9780133457117
  *Paperback*, ISBN10: 0130422681, ISBN13: 9780130422682

- Papoulis A. and S. Pillai, *Probability, Random Variables, and Stochastic Processes*, Fourth edition, McGraw Hill, Inc., 2002.

IDENTIFIERS – *Paperback*, ISBN10: 0071226613, ISBN13: 9780071226615
*Hardback*, ISBN10: 0072817259, ISBN13: 9780072817256

- Proakis J. G. and D. G. Manolakis, *Digital Signal Processing*: *Principles, Algorithms, and Applications*, Pearson New International Edition, Fourth edition, Pearson Education, 2013.

IDENTIFIERS – *Paperback*, ISBN10: 1292025735, ISBN13: 9781292025735

- Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing*: *Concepts and Applications*, Palgrave, Macmillan, 2003.

IDENTIFIERS – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562

See `http://www.see.ed.ac.uk/~{}pmg/SIGPRO`

- Therrien C. W. and M. Tummala, *Probability and Random Processes for Electrical and Computer Engineers*, Second edition, CRC Press, 2011.

IDENTIFIERS – *Hardback*, ISBN10: 1439826986, ISBN13: 978-1439826980

- Press W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Receipes in C*: *The Art of Scientific Computing*, Second edition, Cambridge University Press, 1992.

IDENTIFIERS – *Paperback*, ISBN10: 0521437202, ISBN13: 9780521437202
*Hardback*, ISBN10: 0521431085, ISBN13: 9780521431088

The material in [Kay:1993] and [Papoulis:1991] is covered throughout the course; material in [Therrien:1992] and is covered primarily in the handouts on random processes. The following labelling convention is used for numbering equations that are taken from the various recommended texts. Equations labelled as:

**M:v.w.xyz** are similar to those in [Manolakis:2001] with the corresponding label;

**T:w.xyz** are similar to those in [Therrien:1992] with the corresponding label;

**K:w.xyz** are similar to those in [Kay:1993] with the corresponding label;

**P:v.w.xyz** are used in chapters referring to basic digital signal processing (DSP), and are references made to [Proakis:1996].

# Contents

# Acronyms

| | |
|---|---|
| **AC** | autocorrelation |
| **ACF** | autocorrelation function |
| **AIC** | Akaike's information criterion |
| **AIR** | acoustic impulse response |
| **AR** | autoregressive |
| **ARMA** | autoregressive moving average |
| **BIBO** | bounded-input, bounded-output |
| **BIC** | B-Information criterion |
| **BSS** | blind source separation |
| **CAT** | Parzen's criterion autoregressive transfer function |
| **CPSD** | cross-power spectral density |
| **DC** | "direct current" |
| **DFT** | discrete Fourier transform |
| **DSP** | digital signal processing |
| **DTFT** | discrete-time Fourier transform |
| **FIR** | finite impulse response |
| **FPE** | final prediction error |
| **FS** | Fourier series |
| **FT** | Fourier transform |
| **IDFT** | inverse-DFT |
| **KL** | Karhunen-Loeve |
| **LHS** | left hand side |
| **LS** | least-squares |
| **LSE** | least-squares estimate |
| **LTI** | linear time-invariant |

| | |
|---|---|
| **LTV** | linear time-varying |
| **MA** | moving average |
| **MDL** | minimum description length |
| **ML** | maximum-likelihood |
| **MLE** | maximum-likelihood estimate |
| **MS** | mean-square |
| **MSC** | magnitude square coherence |
| **MSE** | mean-squared error |
| **PSD** | power spectral density |
| **RHS** | right hand side |
| **ROC** | region of convergence |
| **SSS** | strict-sense stationary |
| **WGN** | white Gaussian noise |
| **WSP** | wide-sense periodic |
| **WSS** | wide-sense stationary |
| **cdf** | cumulative distribution function |
| **iff** | if, and only if, |
| **i. i. d.** | independent and identically distributed |
| **pdf** | probability density function |
| **RV** | random variable |
| **w. r. t.** | with respect to |

10;

# 1

# Review of Fourier Transforms and Discrete-Time Systems

This handout will review complex Fourier series and Fourier transforms, followed by a review of discrete-time systems. It covers complex Fourier series, Fourier transforms, Discrete-time Fourier transforms, Discrete Fourier Transforms, Parseval's Theorem, the bilaterial Z-transform, frequency response, and rational transfer functions.

## 1.1 Obtaining the Latest Version of these Handouts

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- This research tutorial is intended to cover a wide range of aspects which cover the fundamentals of statistical signal processing. It is written at a level which assumes knowledge of undergraduate mathematics and signal processing nomenclature, but otherwise should be accessible to most technical graduates.

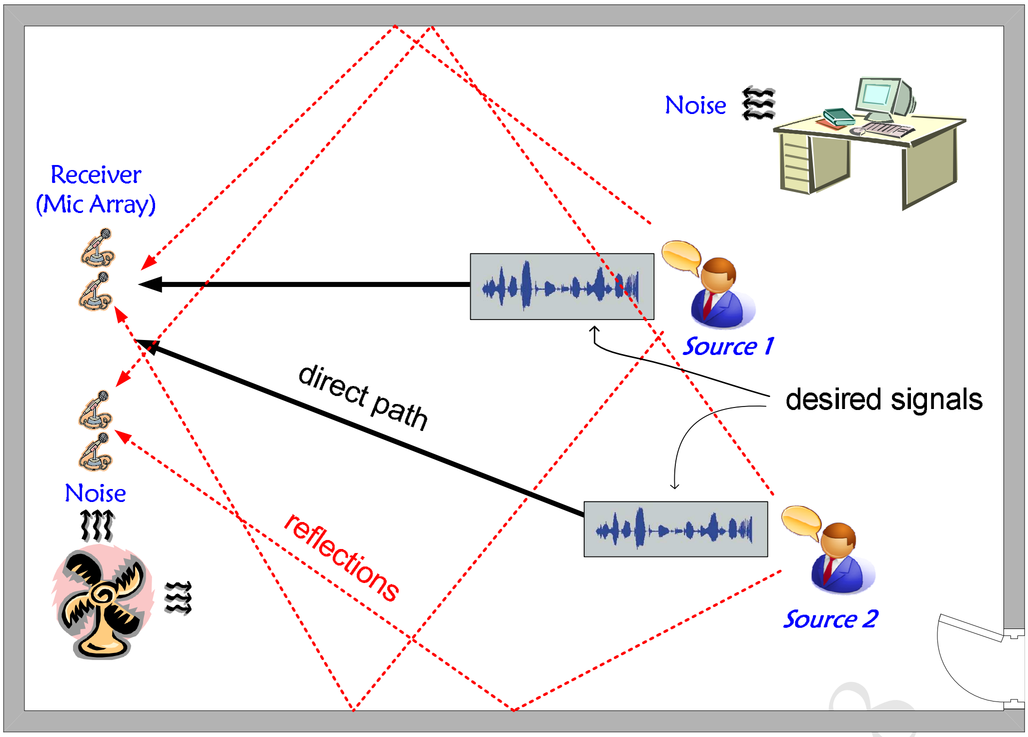



Figure 1.1: Source localisation and BSS. An example of topics using statistical signal processing.

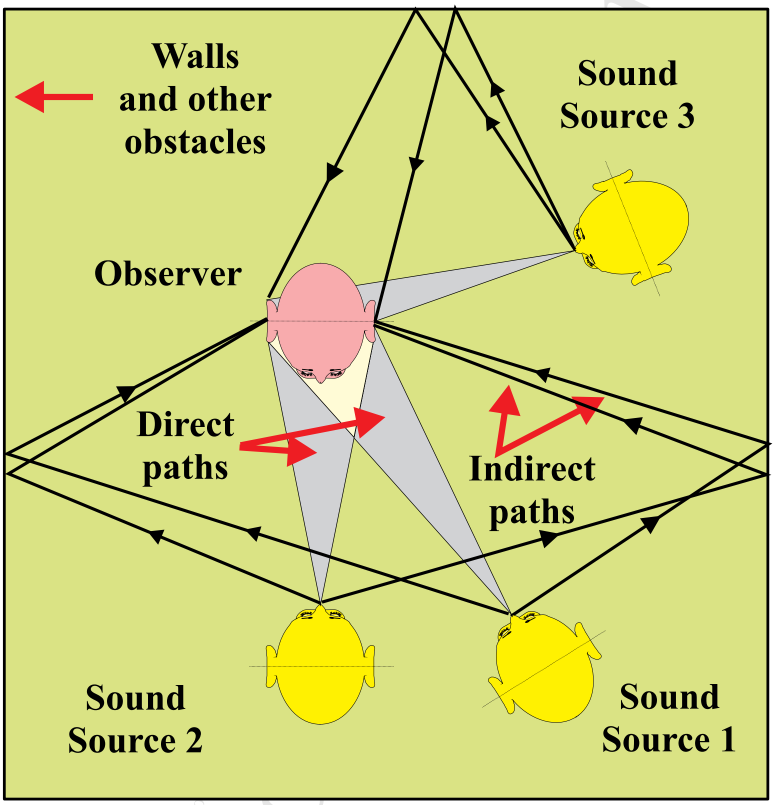



Figure 1.2: Humans turn their head in the direction of interest in order to reduce inteference from other directions; *joint detection, localisation, and enhancement*. An application of probability and estimation theory, and statistical signal processing.

**KEYPOINT! (Latest Slides).** Please note the following:

- This tutorial is being continually updated, and feedback is welcomed. The documents published on the USB stick may differ to the slides presented on the day. In particular, there are likely to be a few typos in the document, so if there is something that isn't clear, please feel free to email me so I can correct it (or make it clearer).
- The latest version of this document can be obtained from the author, Dr James R. Hopgood, by emailing him at: at:

  `mailto:james.hopgood@ed.ac.uk`

  (Update: The notes are no longer online due to the desire to maintain copyright control on the document.)
- Extended thanks are given to the many MSc students over the past 12 years who have helped proof-read and improve these documents.

## 1.2 Introduction

This handout will review complex **Fourier series** and **Fourier transforms**, followed by a review of **discrete-time systems**. The reader is expected to have previously covered most of the concepts in this handout, although it is likely that the reader might need to revise the material if it's been a while since it's been studied. Nevertheless, this revision material is included in the module as review material purely for completeness and reference. It is not intended as a full introduction, although some parts of the review cover the subject in detail.

As discussed in the first handout, if the reader wishes to revise these topics in more detail, the following book comes *highly* recommended:

> Proakis J. G. and D. G. Manolakis, *Digital Signal Processing*: *Principles, Algorithms, and Applications*, Pearson New International Edition, Fourth edition, Pearson Education, 2013.

**IDENTIFIERS** – *Paperback*, ISBN10: 1292025735, ISBN13: 9781292025735

For undergraduate level text books covering signals and systems theory, which it is assumed you have covered, the following book is recommmended:

> Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing*: *Concepts and Applications*, Palgrave, Macmillan, 2003.

**IDENTIFIERS** – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562

See `http://www.see.ed.ac.uk/~{}pmg/SIGPRO`

The latest edition was printed in 2002, but any edition will do. An alternative presentation of roughly the same material is provided by the following book:

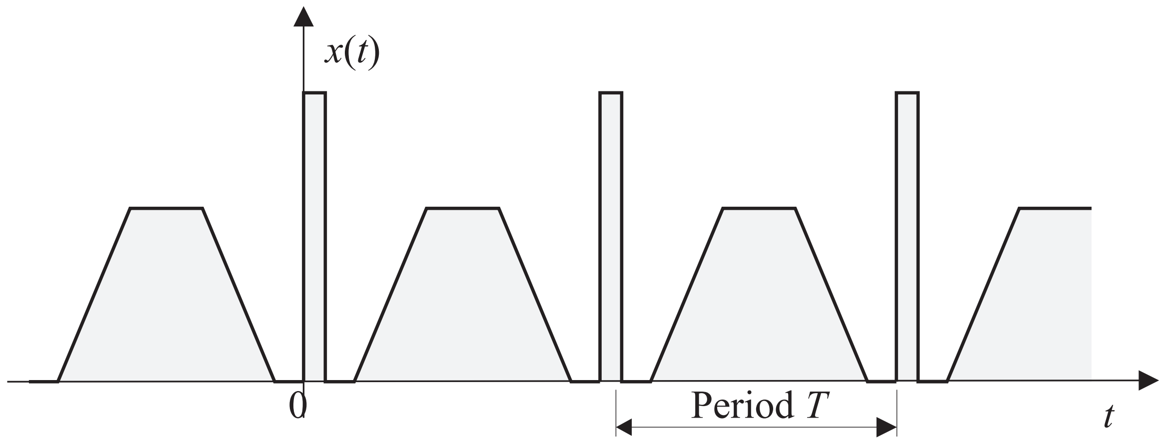


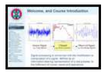

Figure 1.3: An example of a periodic signal with period $T$.

Balmer L., *Signals and Systems: An Introduction*, Second edition, Prentice-Hall, Inc., 1997.

**IDENTIFIERS** – *Paperback*, ISBN10: 0134954729, ISBN13: 9780134956725

In particular, the appendix on complex numbers may prove useful.

## 1.3 Signal Classification



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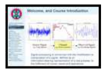
Before considering the analysis of signals and systems, it is necessary to be aware of the general classifications to which signals can belong, and to be aware of the significance of some subtle characteristics that determine how a signal can be analysed. Not all signals can be analysed using a particular technique.

### 1.3.1 Types of signal



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In general, there are four distinct types of signals that must be analysed:

**Continuous-time periodic** Such signals repeat themselves after a fixed length of time known as the period, usually denoted by $T$. This repetition continues ad-infinitum (i.e. forever).

Formally, a signal, $x(t)$, is periodic if

$$x(t) = x(t + m\,T)\,,\ \forall\, m \in \mathbb{Z} \tag{1.1}$$

where the notation $\forall\, m \in \mathbb{Z}$ means that $m$ takes on *all* integer values: in other-words, $m = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$. The smallest positive value of $T$ which satisfies this condition is the defined as the **fundamental period**.

These signals will be analysed using the **Fourier Series**, and are used to represent real-world waveforms that are near to being periodic over a sufficiently long period of time.

An example of a periodic signal is shown in Figure 1.3. This kind of signal vaguely represents a line signal in analogue television, where the rectangular pulses represent line synchronisation signals.

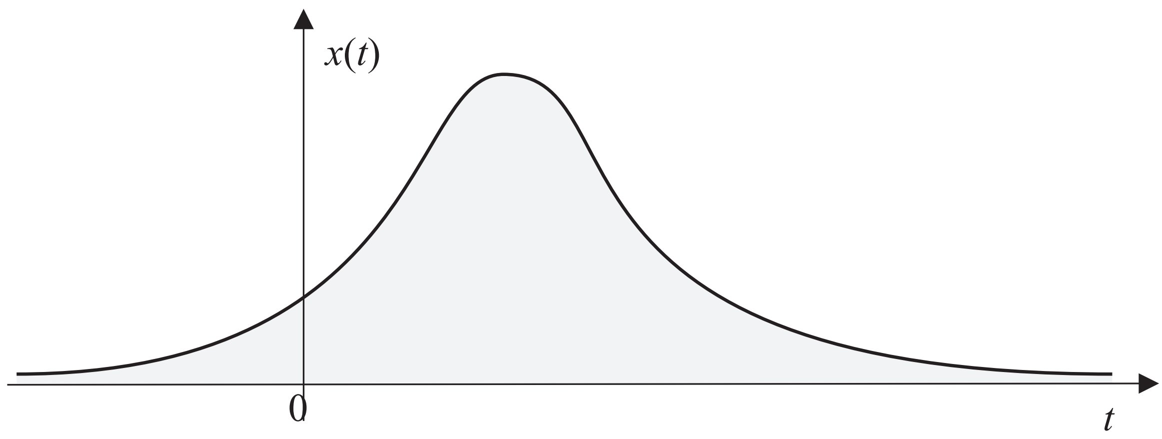



Figure 1.4: An example of an aperiodic signal.

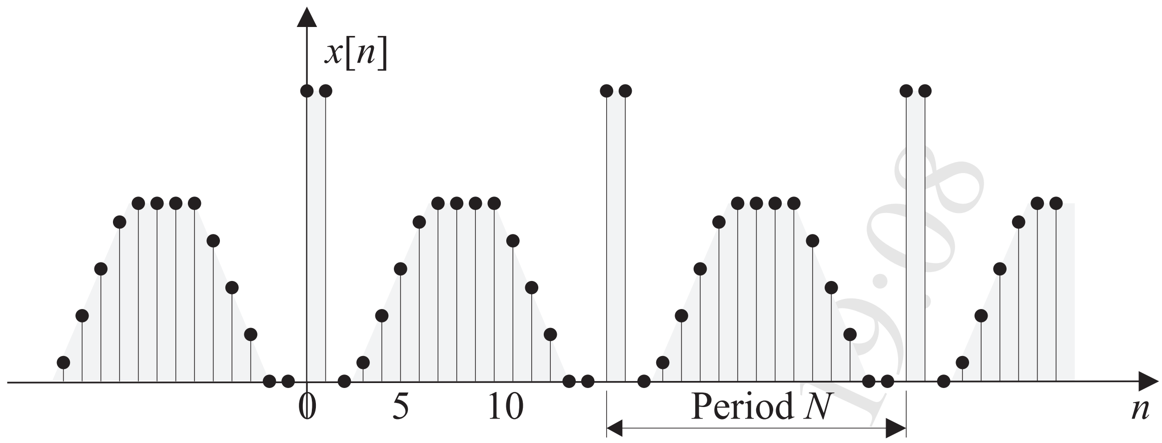



Figure 1.5: A discrete-time periodic signal.

**Continuous-time aperiodic** Continuous-time aperiodic signals are not periodic over all time, although they might be locally periodic over a short time-scale.

These signals can be analysed using the **Fourier transform** for most cases, and more often using the **Laplace transform**. Aperiodic signals are more representative of many real-world signals. Again, real signals don't last for all time, although can last for a considerably long time. An example of an aperiodic signal is shown in Figure 1.4.

**Discrete-time periodic** A discrete-time periodic signal is shown in Figure 1.5, which is essentially a *sampled* version of the signal shown in Figure 1.3. Note in this case, the period is often denoted by $N$, primarily to reflect the fact the time-index is now $n$.

A discrete-time signal, $x[n]$, is periodic if:

$$x[n] = x[n + m\,N]\,,\;\forall m \in \mathbb{Z} \tag{1.2}$$

This is, of course, similar to Equation 1.1.

**Discrete-time aperiodic** Analogous to the continuous-time aperiodic signal in Figure 1.4, a discrete-time aperiodic signal is shown in Figure 1.6.

Aperiodic discrete-time signals will be analysed using the $z$-transform and also the discrete-time Fourier transform (DTFT).

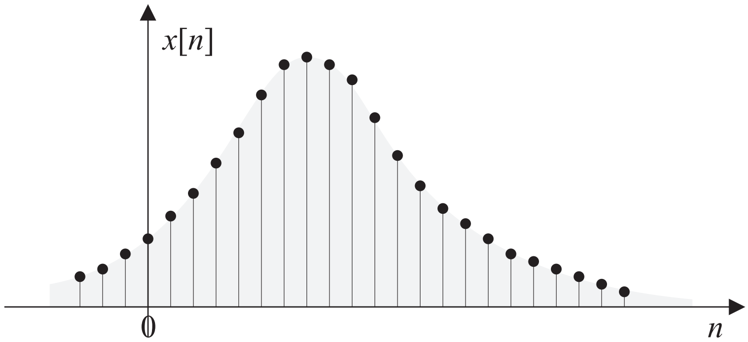



Figure 1.6: An example of a discrete-time aperiodic signal.

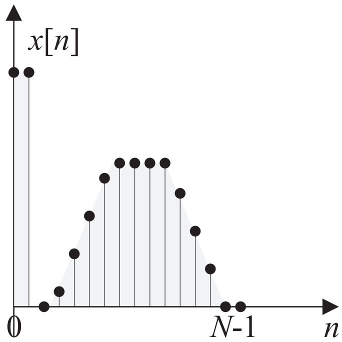
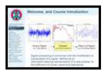



Figure 1.7: An example of a finite-duration signal.

**Finite-length discrete-time signals** Discrete-time signals can also be classified as being finite in length. In other words, they are not assumed to exist for all-time, and what happens outside the **window** of data is assumed unknown. These signals can be modelled using the so-called **discrete Fourier transform (DFT)**.

## 1.3.2 Energy and Power Signals



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As stated in Section 1.3.1, signals can be analysed using a variety of frequency-domain transform methods, such as the **Fourier series**, **Fourier transform**, **Laplace transform**, and for discrete-time, the $z$**-transform**. For example, the Fourier transform is used to analyse aperiodic continuous-time signals.

However, not all aperiodic signals can be analysed using the Fourier transform, and the reason for this can be directly related to a fundamental property of a signal: a measure of *how much signal there is*.

Therefore it is relevant to consider the **energy** or **power** as a means for characterising a signal. The concepts of **power** and **energy** intuitively follow from their use in other aspects of the physical sciences. However, the concept of signals which exist for all time requires careful definitions, in order to determine when it has **energy** and when it has **power**.

Intuitively, energy and power measure *how big* a signal is. A motivating example of measuring the size of something is given in Sidebar 1.

### 1.3.2.1 Motivation for Energy and Power Expressions

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Considering power from an electrical perspective, if a voltage $x(t)$ is connected across a resistance $R$, the dissipated power at time $\tau$ is given by:

**Sidebar 1** Size of a Human Being

Suppose we wish to devise a signal number $V$ as a measure of the size of a human being. Then clearly, the width (or girth) must also be taken into account as well as the height. One could make the simplifying assumption that the shape of a person is a cylinder of variable radius $r$ (which varies with the height $h$). Then one possible measure of the size of a person of height $H$ is the person's volume, given by:

$$V = \pi \int_0^H r^2(h)\, dh \tag{1.3}$$

This can be found by dividing the object into circular discs (which is an approximation), where each disc has a volume $\delta V \approx \pi r^2(h)\, \delta h$. Then the total volume is given by $V = \int dV$.

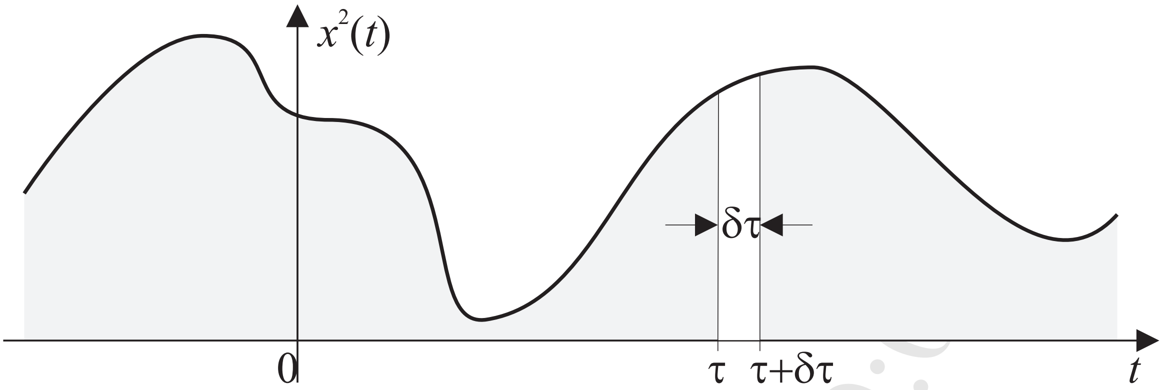



Figure 1.8: Energy Density.

$$P(\tau) = \frac{x^2(\tau)}{R} \propto x^2(\tau) \tag{1.4}$$

where $\propto$ denotes *proportional to*. Since energy and power are related through the expression

$$\text{Energy} = \text{Power} \times \text{Time}, \tag{1.5}$$

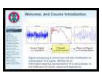
the energy dissipated between times $\tau$ and $\tau + \delta\tau$, as indicated in Figure 1.8, is:

$$\delta E(\tau) \propto P(\tau)\, \delta\tau \propto x^2(\tau)\delta\tau \tag{1.6}$$

The total energy over all time can thus be found by integrating over all time:

$$E \propto \int_{-\infty}^{\infty} x^2(\tau)\, d\tau \tag{1.7}$$

This leads to the formal definitions of energy and power.



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#### 1.3.2.2 Formal Definitions for Energy and Power

Based on the justification in Section 1.3.2.1, the formal abstract definitions for energy and power that are independent of how the energy or power is dissipated are defined below.

**Energy Signals** A continuous-time signal $x(t)$ is said to be an **energy signal** if the total energy, $E$, dissipated by the signal over all time is both *nonzero* and *finite*. Thus:

$$0 < E < \infty \quad \text{where} \quad E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \tag{1.11}$$

**Sidebar 2** Other signal measures

1. While the area under a signal $x(t)$ is a possible measure of its size, because it takes account not only of the amplitude but also of the duration, is defective since even for a very large signal, the positive and negative areas could cancel each other out, indicating a signal of a small size.

2. Using the sum of square values can potentially give undue weighting to any outliers in the signal, where an outlier is defined as an unusual signal variation that is not characteristic of the rest of the signal; an example might be a high-energy shot burst of interference.

3. Therefore, taking the absolute value, $|x(t)| \equiv \operatorname{abs} x(t)$ is a possible measure, and in some circumstances can be used. Unfortunately, dealing with the absolute value of a function can be difficult to manipulate mathematically. However, using the area under the square of the function is not only more mathematically tractable but is also more meaningful when compared with the electrical examples and the volume in Sidebar 1.

4. These notions lead onto the more general subject of **signal norms**. The $L_p$-norm is defined by:

$$L_p(x) \triangleq \left( \int_{-\infty}^{\infty} |x(t)|^p \, dt \right)^{\frac{1}{p}}, \qquad p \geq 1 \tag{1.8}$$

   In particular, the expression for energy is related to the $L_2$-norm, while using the absolute value of the signbal gives rise to the $L_1$-norm:

$$L_1(x) \triangleq \int_{-\infty}^{\infty} |x(t)| \, dt \tag{1.9}$$

   which is the integral of the absolute value as described above in part 3.

5. While Parseval's theorem exists between the time-domain and frequency-domain for relating the $L_2$-norms, in general no relation exists for other values of $p$.

6. Note that the $L_p$-norm generalises for discrete-time signals as follows:

$$L_p(x) \triangleq \left( \sum_{-\infty}^{\infty} |x[t]|^p \right)^{\frac{1}{p}}, \qquad p \geq 1 \tag{1.10}$$

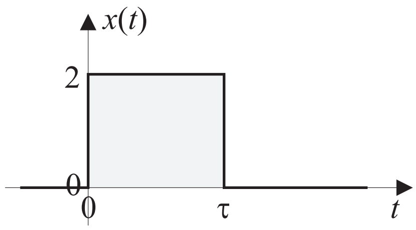



Figure 1.9: Rectangular pulse of length $\tau$.

where $|x(t)|$ means the magnitude of the signal $x(t)$. If $x(t)$ is a real-signal, this is just its amplitude. If $x(t)$ is a complex-signal, then $|x(t)|^2 = x(t)\, x^*(t)$ where $*$ denotes complex-conjugate. In this course, however, only real signals will be encountered.

A necessary condition for the energy to be finite is that the signal amplitude $|x(t)| \to 0$ as $|t| \to \infty$, otherwise the integral in Equation 1.11 will not exist. When the amplitude of $x(t)$ does not tend to zero as $|t| \to \infty$, the signal energy is likely to be infinite. A more meaningful measure of the signal size in such a case would be the time average of the energy if it exists. This measure is called the **power** of the signal.

**Power signals** If the average power delivered by the signal over all time is both *nonzero* and *finite*, the signal is classified as a power signal:

$$0 < P < \infty \quad \text{where} \quad P = \lim_{W \to \infty} \frac{1}{2W} \int_{-W}^{W} |x(t)|^2 \, dt \tag{1.12}$$

where the variable $W$ can be considered as half of the width of a *window* that covers the signal and gets larger and larger.

**Example 1.1.** Name a type of signal which is not an example of an **energy signal**?

SOLUTION. A periodic signal has finite energy over one period, so consequently has infinite energy. However, as a result it has a finite average power and is therefore a power signal, and not an energy signal.

**Example 1.2 (Rectangular Pulse).** What is the energy of the rectangular pulse shown in Figure 1.9 as a function of $\tau$, and for the particular case of $\tau = 4$?

SOLUTION. The signal can be represented by

$$x(t) = \begin{cases} 2 & 0 \le t < \tau \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

so that the square of the signal is also rectangular and given by

$$x^2(t) = \begin{cases} 4 & 0 \le t < \tau \\ 0 & \text{otherwise} \end{cases} \tag{1.14}$$

Since an integral can be interpreted as the area under the curve (see Figure 1.10, the total energy is thus:

$$E = 4\tau \tag{1.15}$$

□

When $\tau = 4$, $E = 16$.

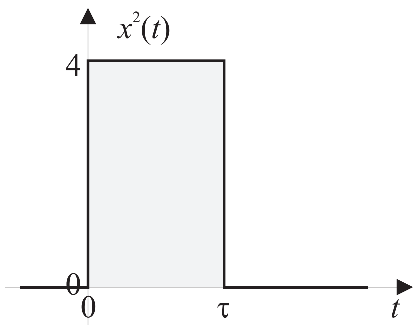
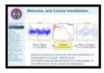



Figure 1.10: The total energy of the signal in Figure 1.9 can be found as the area under the curve representing the square of the rectangular pulse, as shown here.

#### 1.3.2.3 Units of Energy and Power



It is important to consider the physical units associated with energy and power, and therefore to determine how the abstract definitions of $E$ and $P$ in Equation 1.11 and Equation 1.12 can be converted into real energy and power. *New slide*

Consider again power from an electrical perspective. When considering "direct current" (DC) signals, power is given by

$$P_{DC} = \frac{V^2}{R} = \frac{\text{Volts}^2}{\text{Ohms}} = \text{Watts} \tag{1.16}$$

where $V$ is the signal voltage, and $R$ is the resistance through which the power is dissipated. Consider now the units of the abstract definition of power, $P$ in Equation 1.12:

$$P = \frac{1}{\text{time}} \times \text{Volts}^2 \times \text{time} = \text{Volts}^2 = \text{Watts} \times \text{Ohms} \tag{1.17}$$

where the second unit of *time* comes from the integral term $dt$, and in which the integral may be considered as a summation. Therefore, by comparing Equation 1.16 and Equation 1.12, the abstract definition of power, $P$, can be converted to real power by **Ohms** for the case of electrical circuits.

Similarly, the units of energy in Equation 1.11 is given by

$$E = \text{volts}^2 \times \text{time} \tag{1.18}$$

Therefore, to convert the abstract energy to Joules, it is again necessary to divide by **Ohms** by noting that energy is power multiplied by time.

#### 1.3.2.4 Power for Periodic Signals

The expression in Equation 1.12 can be simplified for periodic signals. Consider the periodic signal in Figure 1.3. Note here that there might be confusion with using the same symbol $T$ to mean both the period of a periodic signal and the limit in Equation 1.12. To avoid ambiguity, rewrite Equation 1.12 with $W$ instead of $T$ where $W$ denotes a *window length* over which the power is calculated, and define:

$$P_W = \frac{1}{2W} \int_{-W}^{W} |x(t)|^2 \, dt \tag{1.19}$$

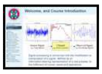
Thus, the average power over two periods is $P_T$, and the average power over $2N$ periods is $P_{NT}$. It should becomes clear that

$$P_T = P_{NT}, \forall N \in \mathbb{Z} \tag{1.20}$$

since the average in each period is the same. Consequently, **power** for a periodic signal with period $T$ may be defined as:

$$P = \frac{1}{T} \int_0^T |x(t)|^2 \, dt \tag{1.21}$$

Note that the limits in Equation 1.21 may be over any period and thus can be replaced by $(\tau, \tau + T)$ for any value of $\tau$.



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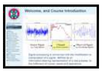
## 1.4 Fourier Series and Fourier Transforms

In this review of Fourier series and transforms, the topics covered are:

- Complex Fourier series
- **Fourier transform**
- The discrete-time Fourier transform
- Discrete Fourier transform



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### 1.4.1 Complex Fourier series

The complex Fourier series is a frequency analysis tool for continuous time periodic signals. Examples of periodic signals encountered in practice include square waves, triangular waves, sawtooth waves, pulse waves and, of course, sinusoids and complex exponentials, as well as half-wave recitifed, full-wave rectified and $p$-phased rectified sinusoids. The basic mathematical representation of periodic signals is the Fourier series, which is a linear weighted sum of harmonically related sinusoids or complex exponentials.

A **periodic continuous-time** deterministic signal, $x_c(t)$, with fundamental period $T_p$ can be expressed as a linear combination of harmonically related complex exponentials:

$$x_c(t) = \sum_{k=-\infty}^{\infty} \check{X}_c(k) \, e^{jk\omega_0 t}, \quad t \in \mathbb{R}, \tag{M:2.2.1}$$

where $\omega_0 = 2\pi F_0 = \frac{2\pi}{T_p}$ is the **fundamental frequency**. Here, $\omega_0$ is in radians per second, and the fundamental frequency, in Hertz, is given by $F_0 = \frac{1}{T_p}$. Moreover,

$$\check{X}_c(k) = \frac{1}{T_p} \int_0^{T_p} x_c(t) \, e^{-jk\omega_0 t} \, dt, \quad k \in \mathbb{Z} \tag{M:2.2.2}$$

are termed the **Fourier coefficients**, or **spectrum** of $x_c(t)$. Note that although the region of integration in Equation M:2.2.2 is from $0$ to $T_p$, it can actually be over any period of the waveform, since the signal, $x_c(t)$, is periodic with period $T_p$.

The $k$th frequency component corresponds to frequency $\omega_k = k\omega_0 = k\frac{2\pi}{T_p}$. The set of exponential functions

$$\mathcal{F}(t) = \{e^{j\omega_0 kt}, \, k \in \mathbb{Z}\} \tag{1.22}$$

can be thought of as the basic *building blocks* from which periodic signals of various types can be constructed with the proper choice of fundamental frequency and Fourier coefficients.

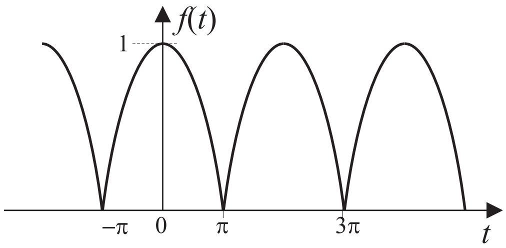



Figure 1.11: Function $f(t)$ of Example 1.3

**Example 1.3 (Complex Fourier Series).** Find the complex form of the Fourier series expansion of the periodic function $f(t)$ defined by:

$$
\begin{aligned}
f(t) &= \cos \frac{1}{2}t \quad (-\pi < t < \pi) \\
f(t+2\pi) &= f(t)
\end{aligned}
\tag{1.23}
$$

SOLUTION. A graph of the function $f(t)$ over the interval $-\pi \leq t \leq 3\pi$ is shown in Figure 1.11. The period $T_p = 2\pi$, so therefore the complex coefficients, denoted by $F_n$, are given by Equation M:2.2.2 as:

$$
F_n = \frac{1}{T_p} \int_0^{T_p} f(t)\, e^{-jn\omega_0 t}\, dt, \quad n \in \mathbb{Z} \tag{1.24}
$$

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \frac{t}{2}\, e^{-jnt}\, dt = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( e^{j\frac{t}{2}} + e^{-j\frac{t}{2}} \right) e^{-jnt}\, dt \tag{1.25}
$$

$$
= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( e^{-j\left(n-\frac{1}{2}\right)t} + e^{-j\left(n+\frac{1}{2}\right)t} \right) dt \tag{1.26}
$$

which, after some trivial integration, gives:

$$
F_n = \frac{1}{4\pi} \left[ \frac{-2e^{-j(2n-1)\frac{t}{2}}}{j(2n-1)} - \frac{2e^{-j(2n+1)\frac{t}{2}}}{j(2n+1)} \right]_{-\pi}^{\pi} \tag{1.27}
$$

$$
= \frac{j}{2\pi} \left[ \left( \frac{e^{-jn\pi}\, e^{j\frac{\pi}{2}}}{2n-1} + \frac{e^{-jn\pi}\, e^{-j\frac{\pi}{2}}}{2n+1} \right) - \left( \frac{e^{jn\pi}\, e^{-j\frac{\pi}{2}}}{2n-1} + \frac{e^{jn\pi}\, e^{j\frac{\pi}{2}}}{2n+1} \right) \right] \tag{1.28}
$$

Noting that $e^{\pm j\frac{\pi}{2}} = \pm j$, and $e^{\pm jn\pi} = \cos n\pi = (-1)^n$, then it follows that:

$$
F_n = \frac{j}{2\pi} \left( \frac{j}{2n-1} - \frac{j}{2n+1} + \frac{j}{2n-1} - \frac{j}{2n+1} \right) (-1)^n \tag{1.29}
$$

$$
= \frac{(-1)^n}{\pi} \left( \frac{1}{2n+1} - \frac{1}{2n-1} \right) = \frac{2(-1)^{n+1}}{(4n^2-1)\pi} \tag{1.30}
$$

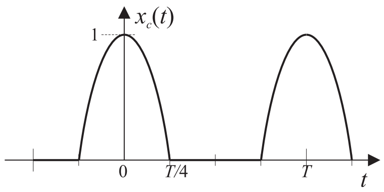
Note that in this case, the coefficients $F_n$ are real. This is expected, since the function $f(t)$ is an even function of $t$. The complex Fourier series expansion for $f(t)$ is therefore:

$$
f(t) = \sum_{n=-\infty}^{\infty} \frac{2(-1)^{n+1}}{(4n^2-1)\pi}\, e^{jnt} \tag{1.31}
$$

□

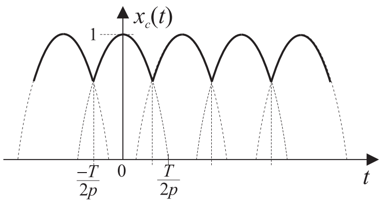
#### 1.4.1.1 Common Fourier Series Expansions

In the following Fourier series expansions, $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency.



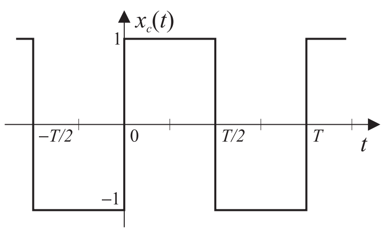

**Half-wave rectified cosine-wave:**

$$x_c(t) = \frac{1}{\pi} + \frac{1}{2}\cos\omega_0 t + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n\omega_0 t)}{4n^2-1}$$



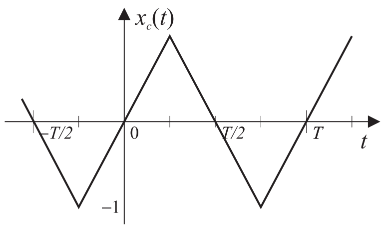

**p-phase rectified cosine-wave ($p \geq 2$):**

$$x_c(t) = \frac{p}{\pi}\sin\frac{\pi}{p}\left[1 + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(pn\omega_0 t)}{p^2n^2-1}\right]$$



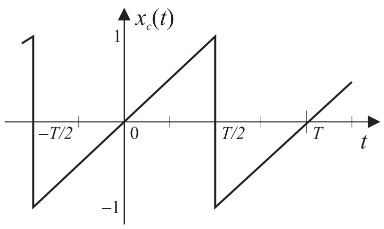

**Square wave:**

$$x_c(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega_0 t}{2n-1}$$



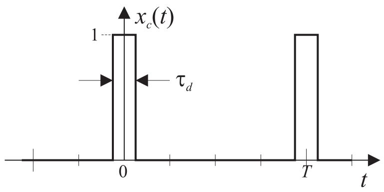

**Triangular wave:**

$$x_c(t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2n-1)\omega_0 t}{(2n-1)^2}$$




**Sawtooth wave:**

$$x_c(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\omega_0 t}{n}$$




**Pulse wave:**

$$x_c(t) = \frac{\tau_d}{T}\left[1 + 2\sum_{n=1}^{\infty}\frac{\sin(n\pi\frac{\tau_d}{T})}{(n\pi\frac{t_d}{T})}\cos(n\omega_0 t)\right]$$

#### 1.4.1.2 Dirichlet Conditions

An important issue that arises in the representation of the continuous time periodic signal $x_c(t)$ by the Fourier series representation,

$$\bar{x}_c(t) = \sum_{k=-\infty}^{\infty}\check{X}_c(k)\,e^{jk\omega_0 t}, \tag{P:4.1.5}$$

is whether or not the series converges for every value of $t \in \mathbb{R}$; i.e., is it true that

$$\bar{x}_c(t) \stackrel{?}{=} x_c(t), \quad \forall t \in \mathbb{R} \tag{1.32}$$

The so-called **Dirichlet conditions** guarantee that the Fourier series converges everywhere except at points of discontinuity. At these points, the Fourier series representation $\bar{x}_c(t)$ converges to the midpoint, or average value, of the discontinuity.

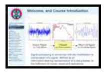
The **Dirichlet conditions** require that the signal $x_c(t)$:

1. has a finite number of discontinuities in any period;
2. contains a finite number of maxima and minima during any period;
3. is **absolutely integrable** in any period; that is:

$$\int_{T_p} |x_c(t)| \, dt < \infty \tag{P:4.1.6}$$

where the integral is over one period. Many periodic signals of practical interest easily satisfy these conditions, and it is reasonable to assume that all practical periodic signals do. However, it is important to beware that pathological cases can make certain proofs or algorithms collapse.

#### 1.4.1.3 Parseval's Theorem (for Fourier series)



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It is sometimes relevant to consider the **energy** or **power** as a means for characterising a signal. These concepts of **power** and **energy** intuitively follow from their use in other aspects of the physical sciences. However, the concept of signals which exist for all time requires careful definitions for when it has **energy** and when it has **power**. Consider the following signal classifications:

**Energy Signals** A signal $x_c(t)$ is said to be an **energy signal** if the total energy, $E$, dissipated by the signal over all time is both *nonzero* and *finite*. Thus:

$$0 < E < \infty \quad \text{where} \quad E = \int_{-\infty}^{\infty} |x_c(t)|^2 \, dt \tag{1.33}$$

**Power signals** If the average power delivered by the signal over all time is both *nonzero* and *finite*, the signal is classified as a power signal:

$$0 < P < \infty \quad \text{where} \quad P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x_c(t)|^2 \, dt \tag{1.34}$$

A periodic signal has infinite energy, but finite average power. The average power of $x_c(t)$ is given by **Parseval's theorem**:

$$P_x = \frac{1}{T_p} \int_0^{T_p} |x_c(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |\check{X}_c(k)|^2 \tag{M:2.2.3}$$

The term $|\check{X}_c(k)|^2$ represents the power in the $k$th frequency component, at frequency $\omega_k = k\frac{2\pi}{T_p}$. Hence,

$$\check{P}_x(k) = |\check{X}_c(k)|^2, \quad -\infty < k < \infty, \, k \in \mathbb{Z} \tag{1.35}$$

is called the **power spectrum** of $x_c(t)$. Consequently, it follows $P_x$ may also be written as:

$$P_x = \sum_{k=-\infty}^{\infty} \check{P}_x(k) \tag{1.36}$$

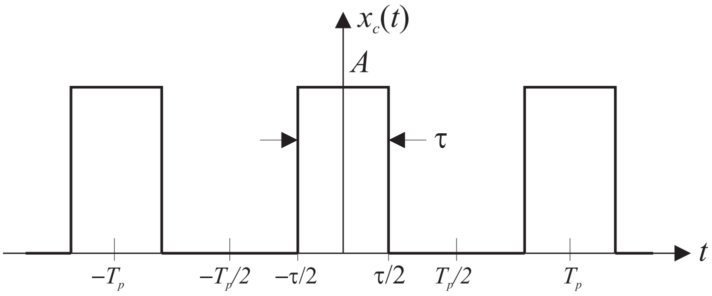



Figure 1.12: Continuous-time periodic train of rectangular pulses.

PROOF. Starting with

$$P_x = \frac{1}{T_p}\int_0^{T_p} x_c(t)\, x_c^*(t)\, dt \tag{1.37}$$

then substituting for the Fourier series expansion of $x_c(t)$ gives:

$$P_x = \frac{1}{T_p}\int_0^{T_p} x_c(t) \left\{\sum_{k=-\infty}^{\infty} \check{X}_c(k)\, e^{jk\omega_0 t}\right\}^* dt \tag{1.38}$$

Noting that the conjugate of a summation (multiplication) is the summation (multiplication) of the conjugates, then:

$$P_x = \frac{1}{T_p}\int_0^{T_p} x_c(t) \sum_{k=-\infty}^{\infty} \check{X}_c^*(k)\, e^{-jk\omega_0 t}\, dt \tag{1.39}$$

Rearranging the order of the integration and the summation gives:

$$P_x = \sum_{k=-\infty}^{\infty} \check{X}_c^*(k) \underbrace{\left\{\frac{1}{T_p}\int_0^{T_p} x_c(t)\; e^{-jk\omega_0 t}(t)\, dt\right\}}_{X_c(k)} \tag{1.40}$$

□

which is the desired result and completes the proof.

Later in this course, the notion of a **power spectrum** will be extended to *stochastic* signals.

**Example 1.4 ( [Proakis:1996, Example 4.1.1, Page 237]).** Determine the Fourier series and the power density spectrum of a rectangular pulse train that is defined over *one* period as follows:

$$x_c(t) = \begin{cases} 0 & \text{if } -\frac{T_p}{2} \le t < -\frac{\tau}{2} \\ A & \text{if } -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0 & \text{if } \frac{\tau}{2} \le t < \frac{T_p}{2} \end{cases} \tag{1.41}$$

where $\tau < T_p$.

SOLUTION. The signal is periodic with fundamental period $T_p$ and, clearly, satisfies the Dirichlet conditions. Consequently, this signal can be represented by the Fourier series. Hence, it follows that

$$\check{X}_c(k) = \frac{1}{T_p}\int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x_c(t)\, e^{-jk\omega_0 t}\, dt = \frac{A}{T_p}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jk\omega_0 t}\, dt \tag{1.42}$$

Two different integrals need to be performed depending on whether $k = 0$ or not. Considering the case when $k = 0$, then the average value of the signal is obtained and given by:

$$\check{X}_c(0) = \frac{1}{T_p} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x_c(t)\, dt = \frac{1}{T_p} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, dt = \frac{A\tau}{T_p} \tag{1.43}$$

For $k \neq 0$, then

$$\check{X}_c(k) = \frac{A}{T_p} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jk\omega_0 t}\, dt = \frac{A}{T_p} \left[ \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \tag{1.44}$$

$$= \frac{A}{jk\omega_0 T_p} \left( e^{jk\omega_0 \frac{\tau}{2}} - e^{-jk\omega_0 \frac{\tau}{2}} \right) = \frac{A\tau}{T_p} \frac{\sin \frac{\tau\omega_0 k}{2}}{k\omega_0 \frac{\tau}{2}} \tag{1.45}$$

$$= \frac{A\tau}{T_p} \operatorname{sinc} \frac{\tau\omega_0 k}{2} \quad \text{where } \operatorname{sinc} x \triangleq \frac{\sin x}{x} \tag{1.46}$$

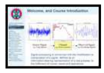
Hence, the power density spectrum for the rectangular pulse is:

$$\left| \check{X}_c(k) \right|^2 = \left( \frac{A\tau}{T_p} \right)^2 \operatorname{sinc}^2 \frac{\tau\omega_0 k}{2}, \quad k \in \mathbb{Z} \tag{P:4.1.19}$$

□

where it is noted that $\operatorname{sinc}(0) = 1$.

## 1.4.2 Fourier transform



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An **aperiodic continuous-time** deterministic signal, $x_c(t)$, can be expressed in the frequency domain using the **Fourier transform** pairs:

$$x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\omega)\, e^{j\omega t}\, d\omega \tag{M:2.2.5}$$

and

$$X_c(\omega) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j\omega t}\, dt \tag{M:2.2.4}$$

$X_c(\omega)$ is called the **spectrum** of $x_c(t)$. Again, note that [Manolakis:2000] uses the defintion $F = \frac{\omega}{2\pi}$. Continuous-time aperiodic signals have continuous aperiodic spectra.

There are a few mathematical requirements that $x_c(t)$ must satisfy for $X_c(\omega)$ to exist; these can be summarised by the phrase that *the signal must be well-behaved*. More specifically, the set of conditions that guarantee the existence of the Fourier transform are the Dirichlet conditions which are the same as for Fourier series.

**Example 1.5 (Fourier Transforms).** Find the Fourier transform of the one-sided exponential function

$$f(t) = H(t)\, e^{-at} \quad \text{where } a > 0 \tag{1.47}$$

and where $H(t)$ is the Heaviside unit step function show in Figure 1.13 and given by:

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.48}$$

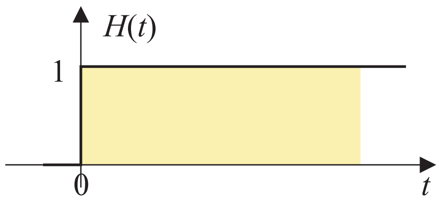



Figure 1.13: The Heaviside step function $H(t)$.

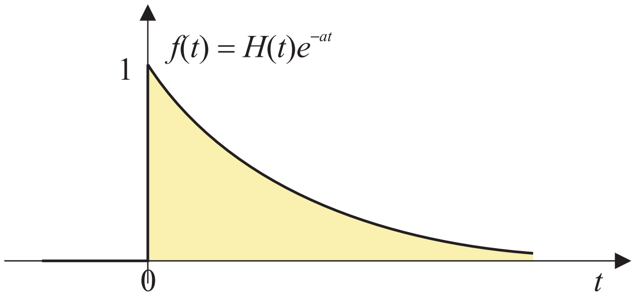



Figure 1.14: Exponential decaying function, $f(t) = H(t)\, e^{-at}$ for $a > 0$.

SOLUTION. Since $f(t) \to 0$ as $t \to \infty$, then the signal energy is bounded, as indicated by plotting the graph of $f(t)$ as shown in Figure 1.14.

A Fourier transform therefore exists, and is given by:

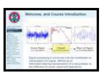
$$X_c(\omega) = \int_{-\infty}^{\infty} H(t)\, e^{-at}\, e^{-j\omega t}\, dt \tag{1.49}$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt = \left[ -\frac{e^{-(a+j\omega)t}}{a+j\omega} \right]_0^{\infty} \tag{1.50}$$

giving

$$X_c(\omega) = \frac{1}{a+j\omega}, \qquad \text{for } -\infty < \omega < \infty \tag{1.51}$$

□



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#### 1.4.2.1 Parseval's theorem (for Fourier transforms)

The *energy* of $x_c(t)$ is, as for **Fourier series**, computed in either the time or frequency domain by **Parseval's theorem**:

$$E_x = \int_{-\infty}^{\infty} |x_c(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_c(\omega)|^2\, d\omega \tag{M:2.2.7}$$

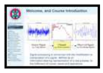
The function $|X_c(\omega)|^2 \geq 0$ shows the distribution of energy of $x_c(t)$ as a function of frequency, $\omega$, and is called the **energy spectrum** of $x_c(t)$.

PROOF. The derivation of Parseval's theorem for Fourier transforms follows a similar line to the derivation of Parseval's theorem for Fourier series; it proceeds as follows:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} x_c(t)\, x_c^\star(t)\, dt = \int_{-\infty}^{\infty} x_c(t)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c^\star(\omega)\, e^{-j\omega t}\, d\omega\; dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c^\star(\omega) \int_{-\infty}^{\infty} x_c(t)\; e^{-j\omega t}\; dt\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c^\star(\omega)\, X_c(\omega)\, d\omega \end{aligned} \tag{1.52}$$

□

### 1.4.3 The discrete-time Fourier transform



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Turning to discrete-time deterministic signals, the natural starting point is to consider aperiodic signals that exist for all discrete-time; i.e. $\{x[n]\}_{-\infty}^{\infty}$. It is interesting to note that there are fewer convergence issues with transforms for discrete-time signals than there are in the continuous-time case.

An **aperiodic discrete-time** deterministic signal, $\{x[n]\}_{-\infty}^{\infty}$, can be synthesised from its **spectrum** using the inverse-discrete-time Fourier transform, given by:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega T}\right) e^{j\omega n}\, d\omega, \quad n \in \mathbb{Z} \tag{M:2.2.13}$$

and the discrete-time Fourier transform (DTFT):

$$X\left(e^{j\omega T}\right) = \sum_{\text{all } n} x[n]\, e^{-j\omega n}, \quad \omega \in \mathbb{R} \tag{M:2.2.12}$$

$X\left(e^{j\omega T}\right)$ is the **spectrum** of $x[n]$.

Since $X\left(e^{j\omega T}\right) = X\left(e^{j(\omega+2\pi k)}\right)$, discrete-time aperiodic signals have continuous periodic spectra with **fundamental period** $2\pi$. However, this property is just a consequence of the fact that the frequency range of any discrete-time signal is limited to $[-\pi, \pi)$ or $[0, 2\pi)$; any frequency outside this interval is equivalent to some frequency within this interval.

There are two basic differences between the Fourier transform of a discrete-time finite-energy aperiodic signal, as represented by the discrete-time Fourier transform, and the Fourier transform of a finite-energy continuous-time aperiodic signal:

1. For continuous-time signals, the Fourier transform, and hence the spectrum of the signal, have a frequency range of $(-\infty, \infty)$. In contrast, the frequency range for a discrete-time signal is unique over the frequency range $[-\pi, \pi)$ or, equivalently, $[0, 2\pi)$.

2. Since $X\left(e^{j\omega T}\right)$ in the DTFT is a periodic function of frequency, it has a Fourier series expansion, provided that the conditions for the existence of the Fourier series are satisfied. In fact, from the fact that $X\left(e^{j\omega T}\right)$ is given by the summation of exponentially weighted versions of $x[n]$ is is clear that the DTFT already has the form of a Fourier series. This is not true for the Fourier transform.

In order for $X\left(e^{j\omega T}\right)$ to exist, $x[n]$ must be absolutely summable:

$$\sum_{\text{all } n} |x[n]| < \infty \tag{M:2.2.11}$$

Finally, as for the Fourier series, and the Fourier transform, discrete-time aperiodic signals have energy which satisfies Parseval's theorem:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X\left(e^{j\omega T}\right)\right|^2 d\omega \tag{P:4.2.41}$$

### 1.4.4 Discrete Fourier transform

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Any finite-length or **periodic discrete-time** deterministic signal, $\{x[n]\}_0^{N-1}$, can be written by the Fourier series, or inverse-DFT (IDFT):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N} nk}, \quad n \in \mathcal{N} \tag{M:2.2.8}$$

where $\mathcal{N} = \{0, 1, \dots, N-1\} \subset \mathbb{Z}^+$, and where the DFT:

$$X_k = \sum_{n=0}^{N-1} x[n]\ e^{-j\frac{2\pi}{N}nk}, \quad k \in \mathcal{N} \tag{M:2.2.9}$$

are the corresponding Fourier coefficients. The sequence $X_k$, $k \in \mathbb{R}$ is the **spectrum** of $x[n]$. $X_k$ is discrete and periodic with the same period as $x[n]$.

Note that a finite-length discrete-time signal of length $N$ has the same Fourier transform, through the DFT, as an infinite-length discrete-time periodic signal of period $N$. Hence, these equivalent perspectives will be considered synonymous.

PROOF (DERIVATION OF DISCRETE FOURIER TRANSFORM). If the **discrete-time** signal $x[n]$ is **periodic** over $N$ samples, then it can be written over one period in continuous time as:

$$x_c(t) = T_p \sum_{n \in \mathcal{N}} x[n]\ \delta(t - n\,T_s)\,, \quad 0 \le t < T_p \tag{1.53}$$

where $\mathcal{N} = \{0, \dots, N-1\}$, $T_s$ is the sampling period, and $T_p = N\,T_s$ is the period of the process. Substituting into the expression for the **Fourier series**, using the **sifting property** and noting that $\omega_0 = \frac{2\pi}{T_p} = \frac{2\pi}{NT_s}$, gives:

$$X_k = \frac{1}{T_p} \int_0^{T_p} x_c(t)\ e^{-jk\omega_0 t}\,dt \tag{1.54}$$

$$= \frac{1}{T_p} \int_0^{T_p} \left\{ T_p \sum_{n \in \mathcal{N}} x[n]\ \delta(t - n\,T_s) \right\} e^{-jk\omega_0 t}\,dt \tag{1.55}$$

$$= \sum_{n \in \mathcal{N}} x[n] \int_0^{T_p} \delta(t - n\,T_s)\, e^{-jk\omega_0 t}\,dt \tag{1.56}$$

$$= \sum_{n \in \mathcal{N}} x[n]\ e^{-j\frac{2\pi}{N}nk} \tag{1.57}$$

□

The IDFT can be obtained using the appropriate identities to ensure a unique inverse.

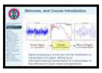
#### 1.4.4.1 Parseval's Theorem for Finite Length Discrete-Time Signals

The average power of a finite length or periodic discrete-time signal with period $N$ is given by

$$P_x = \sum_{n=0}^{N-1} |x[n]|^2 \tag{P:4.2.10}$$

Through the same manipulations as for Parseval's theorems in the cases presented above, which are left as an exercise for the reader, it is straightforward to show that:

$$P_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_k|^2 \tag{P:4.2.11}$$



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#### 1.4.4.2 The DFT as a Linear Transformation

The formulas for the DFT and IDFT may be expressed as:

$$X_k = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}, \quad k \in \mathcal{N} \tag{P:5.1.20}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k\, W_N^{-nk}, \quad n \in \mathcal{N} \tag{P:5.1.21}$$

where, by definition:

$$W_N = e^{-j\frac{2\pi}{N}} \tag{P:5.1.22}$$

which is the $N$th root of unity. Note here that, if $W_N$ has been pre-calculated, then the computation of each point of the DFT can be accomplished by $N$ complex multiplications and $N-1$ complex additions. Hence, the $N$-point $DFT$ can be computed in a total of $N^2$ complex multiplications and $N(N-1)$ complex additions.

It is instructive to view the DFT and IDFT as linear transformations on the sequences $\{x[n]\}_0^{N-1}$ and $\{X_k\}_0^{N-1}$. Defining the following vectors and matrices:

$$\mathbf{x}_N = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad \mathbf{X}_N = \begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix} \tag{1.58}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{1.59}$$

Observe that $X_k$ can be obtained by the inner-product of the $(k-1)$ th-order row with the column $\mathbf{x}_N$:

$$X_k = \begin{bmatrix} 1 & W_N^k & W_N^{2k} & \cdots & W_N^{(N-1)k} \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} \tag{1.60}$$

Then the $N$-point DFT may be expressed in vector-matrix form as:

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{P:5.1.24}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. Observe that $\mathbf{W}_N$ is a symmetric matrix. Assuming that the inverse of $\mathbf{W}_N$ exists, then Equation P:5.1.24 can be inverted by pre-multiplying both sides by $\mathbf{W}_N^{-1}$, to obtain:

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \tag{P:5.1.25}$$

This is the expression for the IDFT, which can also be expressed in matrix form as:

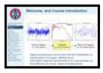
$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N \tag{P:5.1.26}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$. Hence, it follows that:

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^* \quad \text{or} \quad \mathbf{W}_N \mathbf{W}_N^* = N \mathbf{I}_N \tag{P:5.1.27}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix. Hence, $\mathbf{W}_N$ is an orthogonal or unity matrix.

#### 1.4.4.3 Properties of the discrete Fourier transforms



There are some important basic properties of the DFT that should be noted. The notation used to *New slide* denote the $N$-point DFT pair $x[n]$ and $X_k$ is

$$x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k \tag{1.61}$$

**Periodicity** If $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$, then:

$$x[n+N] = x[n] \quad \text{for all } n \tag{P:5.2.4}$$

$$X_{k+N} = X_k \quad \text{for all } k \tag{P:5.2.5}$$

These periodicities in $x[n]$ and $X_k$ follow immediately from the definitions of the DFT and IDFT.

**Linearity** If $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$ and $y[n] \overset{\text{DFT}}{\rightleftharpoons} Y_k$, then

$$\alpha_1 x[n] + \alpha_2 y[n] \overset{\text{DFT}}{\rightleftharpoons} \alpha_1 X_k + \alpha_2 Y_k \tag{P:5.2.6}$$

for any real or complex-valued constants $\alpha_1$ and $\alpha_2$.

**Symmetry of real-valued sequences** If the sequence $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$ is real, then

$$X_{N-k} = X_k^* = X_{-k} \tag{P:5.2.24}$$

**Complex-conjugate properties** If $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$ then

$$x^*[n] \overset{\text{DFT}}{\rightleftharpoons} X_{N-k}^* \tag{P:5.2.45}$$

PROOF. The DFT of the sequence $x[n]$ is given by:

$$X_k = \sum_{n=0}^{N-1} x[n]\; e^{-j\frac{2\pi}{N}nk}, \quad k \in \mathcal{N} \tag{M:2.2.9}$$

and the DFT of $y[n] = x^*[n]$ is given by:

$$Y_k = \sum_{n=0}^{N-1} x^*[n]\; e^{-j\frac{2\pi}{N}nk} \tag{1.62}$$

Taking complex conjugates, and noting that $e^{j\frac{2\pi}{N}mk} = e^{-j\frac{2\pi}{N}m(N-k)}$, then:

$$Y_k^* = \sum_{n=0}^{N-1} x[n]\; e^{-j\frac{2\pi}{N}m(N-k)} = X_{N-k} \tag{1.63}$$

□

Hence, giving $x^*[n] \overset{\text{DFT}}{\rightleftharpoons} X_{N-k}^*$ as required.

**Time reversal of a sequence** If $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$ then

$$x[N-n] \overset{\text{DFT}}{\rightleftharpoons} X_{N-k} \tag{P:5.2.42}$$

Hence, reversing the $N$-point sequence in time is equivalent to reversing the DFT values in frequency.

PROOF. From the definition of the DFT, if $y[n] = x[N-n]$, then:

$$Y_k = \sum_{n=0}^{N-1} x[N-n]\; e^{-j\frac{2\pi}{N}\,nk} = \sum_{m=1}^{N} x[m]\; e^{-j\frac{2\pi}{N}\,(N-m)k} \tag{1.64}$$

where the second summation comes from changing the index from $n$ to $m = N - n$. Noting then, that $x[N] = x[0]$,then this may be written as

$$Y_k = \sum_{m=0}^{N-1} x[m]\; e^{-j\frac{2\pi}{N}\,(N-m)k} = \sum_{m=0}^{N-1} x[m]\; e^{j\frac{2\pi}{N}\,mk} \tag{1.65}$$

$$= \sum_{m=0}^{N-1} x[m]\; e^{-j\frac{2\pi}{N}\,m(N-k)} = X_{N-k} \tag{1.66}$$

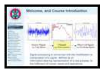
□

as required.

**Circular Convolution** As with many linear transforms, convolution in the time-domain becomes multiplication in the frequency domain, and vice-versa. Since the signals are periodic, it is necessary to introduce the idea of circular convolution. Details of this are discussed in depth in [Proakis:1996, Section 5.2.2, Page 415] and are currently ommitted here. However, assuming that convolution is interpreted in the circular sense (i.e. taking advantage of the periodicity of the time-domain signals), then if $x[n] \overset{\text{DFT}}{\rightleftharpoons} X_k$ and $y[n] \overset{\text{DFT}}{\rightleftharpoons} Y_k$, then:

$$x[n] * y[n] \overset{\text{DFT}}{\rightleftharpoons} X_k\, Y_k \tag{P:5.2.41}$$

## 1.5 Review of discrete-time systems



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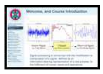
The following aspects of **discrete-time systems** are reviewed:

- Basic discrete-time signals
- The $z$-transform
- Review of **linear time-invariant** systems
- Rational **transfer functions**

### 1.5.1 Basic discrete-time signals



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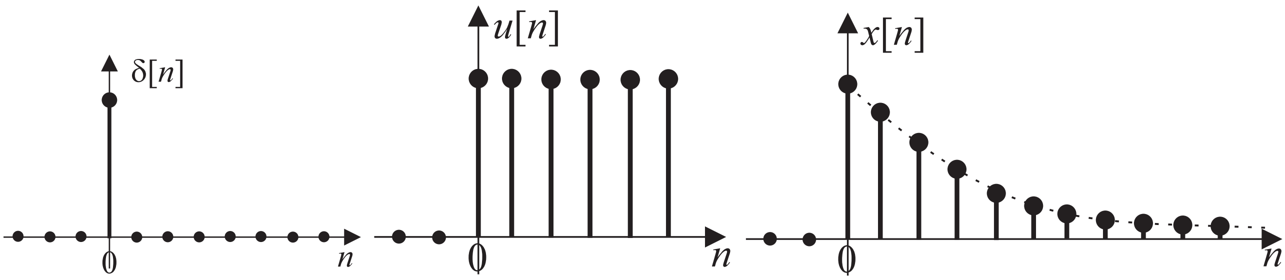
In general, the notation $x[n]$ is used to denote a sequence of numbers that represent a discrete-time signal. The $n$th sample refers to the value of this sequence for a specific value of $n$. In a strict sense, this terminology is only correct if the discrete-time signal has been obtained by sampling a continuous-time signal $x_c(t)$. In the case of periodic sampling with sampling period $T$, then $x[n] = x_c(nT)\,,\ n \in \mathbb{Z}$; that is, $x[n]$ is the $n$th sample of $x_c(t)$.

There are some basic discrete-time signals that will be used repeatedly throughout the course, and these are shown in Figure 1.15:




(a) The unit sample or unit impulse sequence $\delta[n]$.

(b) The unit step sequence $u[n]$.

(c) The exponential decay sequence.

Figure 1.15: Basic discrete-time signals.

1. The **unit sample** or **unit impulse** sequence $\delta[n]$ is defined as:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{M:2.1.1}$$

2. The **unit step** sequence, $u[n]$ is defined as:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{M:2.1.2}$$

3. The **exponential sequence** is of the form

$$x[n] = a^n, \quad -\infty < n < \infty,\, n \in \mathbb{Z} \tag{M:2.1.3}$$

If $a$ is a complex number, such that $a = r\, e^{j\omega_0}$ for $r > 0$, $\omega_0 \neq 0, \pi$, then $x[n]$ is complex valued and given by:

$$x[n] = r^n\, e^{j\omega_0 n} = x_R[n] + jx_I[n] \tag{M:2.1.4}$$
$$= r^n \cos\omega_0 n + jr^n \sin\omega_0 n \tag{1.67}$$

where $x_R[n]$ and $x_I[n]$ are real sequences given by:

$$x_R[n] = r^n \cos\omega_0 n \quad \text{and} \quad x_I[n] = r^n \sin\omega_0 n \tag{M:2.1.5}$$

4. The **critical decay sequence** is of the form

$$x[n] = a\, n\, r^n\, u[n]\,, \quad n \in \mathbb{Z} \tag{1.68}$$

which is discussed further in Sidebar 3.

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### 1.5.2 The $z$-transform

The $z$-transform of a sequence is a very powerful tool for the analysis of discrete linear and time-invariant systems; it plays the same role in the analysis of discrete-time signals and linear time-invariant (LTI) systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems. For example, as will be seen, in the $z$-domain, also known as the complex $z$-plane,

**Sidebar 3** The signal $n\,r^n$

The discrete-time signal

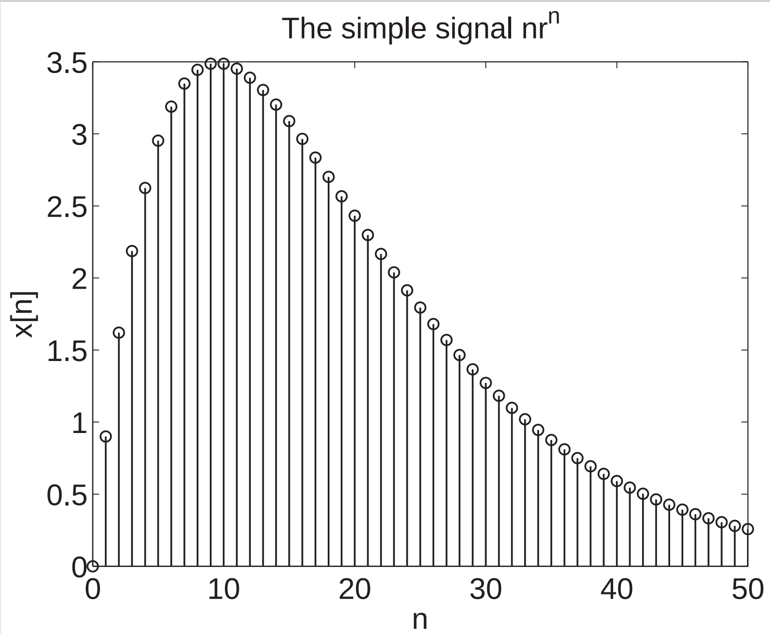
$$x[n] = a\,n\,r^n \tag{1.69}$$

is equivalent to the continuous-time signal $x[t] = t\,e^{-\alpha t}$, and both are important, as they represent the response of a **critically damped system**, as will be seen later. Note in both cases that:

$$\lim_{n\to\infty} n\,r^n \to 0 \tag{1.70}$$

The shape of $x[n]$ is shown below for $r = 0.9$, and note the relationship derived in Sidebar 4 that:

$$n\,r^n \overset{z^+}{\rightleftharpoons} \frac{r}{(1-r)^2} \quad \text{if } |r| < 1 \tag{1.71}$$



the convolution of two time-domain signals is equivalent to multiplication of their corresponding $z$-transforms. This property greatly simplifies the analysis of the response of an LTI system to various inputs.

Although the Fourier transform also satisfies the property that convolution in the time domain becomes multiplication in the frequency domain, it is not always possible to calculate the Fourier transform of a signal, $x[n]$, even for some elementary signals that are important for the analysis of systems. For example, if $x[n]$ is a **power signal** (having finite power), rather than an **energy signal**, the discrete-time Fourier transform (DTFT) does not exist.

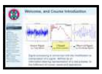
One such signal, of practical importance, is the unit step function, $u[t]$, as can be illustrated by attempting to evaluate the DTFT:

$$U\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n} \tag{1.72}$$

This is a geometric series, of the form $\sum_{n=0}^{\infty} r^n$ where $r = e^{-j\omega}$; however, this series *diverges* since $|r| = 1$. Therefore, the DTFT does not exist; this could also have been deduced from the fact that $u[n]$ is not absolutely summable, which a necessary condition for a Fourier transform to exist:

$$\sum_{\text{all } n} |u[n]| = \sum_{n=0}^{\infty} 1 \not< \infty \tag{1.73}$$

The solution is to multiply the signal by a convergence factor, which leads to the $z$-transform. Details of the derivation can be found in some text books.



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### 1.5.2.1 Bilateral $z$-transform

The bilateral $z$-transform is defined by the following pairs of equations:

$$X(z) \triangleq \mathcal{Z}[x[n]] = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \tag{M:2.2.29}$$

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz \tag{M:2.2.30}$$

where $z$ is a complex variable. This is usually denoted as:

$$x[n] \stackrel{z}{\rightleftharpoons} X(z) \quad \text{or} \quad X(z) = \mathcal{Z}[x[n]] \tag{1.74}$$

The set of values of $z$ for which the power series in the (direct) $z$-transform converges is called the region of convergence (ROC) of $X(z)$. A sufficient condition for convergence is:

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z^{-n}| < \infty \tag{M:2.2.31}$$

The unilateral or one-sided $z$-transform, which is more commonly encountered in undergraduate Engineering courses, is discussed below in Section 1.5.2.3. For the moment, it suffices to mention that the difference between them usually comes down to the initial conditions, and therefore a discussion of the bilateral transform is not too restrictive at this point.

**Sidebar 4** The Ubiquitous Geometric Progression

The **geometric progresson** occurs frequently in discrete-time analysis due to the existance of the summation operator and the commonality of exponential decay functions. It is essentially the discrete-time equivalent of integrating an exponential function. The geometric progression is given by

$$\sum_{n=0}^{N} a\, r^n = a \frac{1-r^{N+1}}{1-r} \tag{1.75}$$

$$\sum_{n=0}^{\infty} a\, r^n = a \frac{1}{1-r} \quad \text{if } |r| < 1 \tag{1.76}$$

More interesting are variants of the geometric progression that can be obtained by simple manipulations, such as differentiating both sides of Equation 1.76 with respect to (w. r. t.) $r$:

$$\frac{d}{dr}\left[\sum_{n=0}^{\infty} a\, r^n\right] = \frac{d}{dr}\left[a \frac{1}{1-r}\right] \tag{1.77}$$

$$\sum_{n=0}^{\infty} a\, n\, r^{n-1} = a \frac{1}{(1-r)^2} \tag{1.78}$$

or, multiplying both sides by $r$, gives:

$$\sum_{n=0}^{\infty} a\, n\, r^n = a \frac{r}{(1-r)^2} \quad \text{if } |r| < 1 \tag{1.79}$$

which is also a useful identity. The signal $x[n] = n\, r^n$ is an important one and discussed further in Sidebar 3. Differentiating repeated times gives a general expresion for $\sum n^p\, r^n$ which can often be useful.

By evaluating the $z$-transform on the unit circle of the $z$-plane, such that $z = e^{j\omega}$, then:

$$X(z)\big|_{z=e^{j\omega}} = X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \tag{M:2.2.32}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega T}\right) e^{j\omega n}\, d\omega \tag{M:2.2.33}$$

which are the DTFT and inverse-DTFT relating the signals $x[n]$ and $X\left(e^{j\omega T}\right)$. This relation holds only if the unit circle is inside the ROC.

**Example 1.6 ( [Proakis:1996, Example 3.1.3, Page 154]).** Determine the $z$-transform of the signal:

$$x[n] = \alpha^n\, u[n] \equiv \begin{cases} \alpha^n & n \ge 0 \\ 0 & n < 0 \end{cases} \tag{1.80}$$

SOLUTION. From the definition of the $z$-transform, it follows that:

$$X(z) = \sum_{k=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\alpha\, z^{-1}\right)^n \tag{1.81}$$

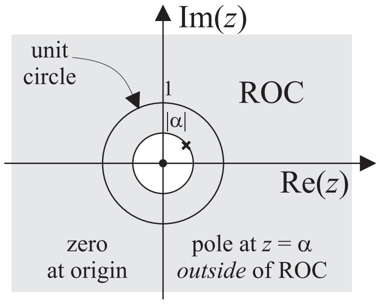


Figure 1.16: The region of convergence (ROC) for the transfer function in Equation P:3.1.7.

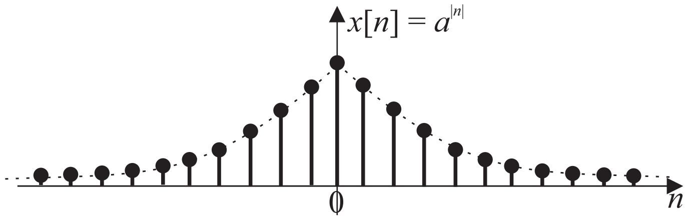


Figure 1.17: The sequence $x[n] = a^{|n|}$.

The summation on the right is a geometric progression, and converges to $\frac{1}{1-\alpha z^{-1}}$ if, and only if, (iff) $|\alpha z^{-1}| < 1$ or, equivalently, $|z| > |\alpha|$. Further details on the geometric progression are given in Sidebar 4. Thus, this gives the $z$-transform pair:

$$x[n] = \alpha^n \, u[n] \overset{z}{\rightleftharpoons} X(z) = \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |z| > |\alpha| \tag{P:3.1.7}$$

Note that, in general, $\alpha$ need not be real. The ROC is the exterior of a circle having radius $|\alpha|$. The ROC is shown in Figure 1.16. The $z$-transform in Equation P:3.1.7 may be written as:

$$X(z) = \frac{z}{z - \alpha} \quad \text{ROC: } |z| > |\alpha| \tag{1.82}$$

□

and therefore it has a pole at $z = \alpha$ and a zero at $z = 0$. The position of the pole is outside the ROC, which is as expected since the $z$-transform does not converge at a pole; it tends to infinity instead. However, simply because there is a zero at the origin does not mean the $z$-transform converges at that point – it doesn't, since it is outside of the ROC. However, the concept of the zero is still important and is thus still drawn on the pole-zero diagram.

**Example 1.7 (Two-sided exponential (Laplacian exponential)).** What is the bilateral $z$-transform of the sequence $x[n] = a^{|n|}$ for all $n$ and some real constant $a$, where $|a| < 1$?

SOLUTION. The bilateral $z$-transform of a sequence $x[n] = a^{|n|}$, shown in Figure 1.17, is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \, z^{-n} = \sum_{n=-\infty}^{\infty} a^{|n|} \, z^{-n} \tag{1.83}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} \, z^{-n} + \sum_{n=0}^{\infty} a^n \, z^{-n} \tag{1.84}$$

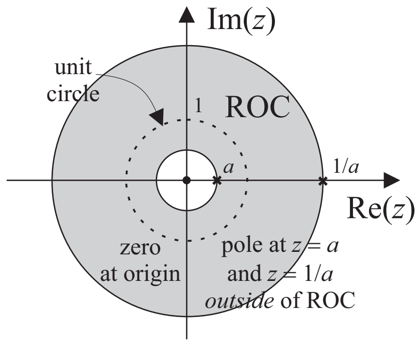


Figure 1.18: The region of convergence (ROC) for the transfer function in Equation 1.88.

Setting $m = -n$ in the first summation, noting that when $n = -\infty$ then $m = \infty$, and when $n = 0$ then $m = 0$, gives:

$$X(z) = \sum_{n=1}^{\infty} (az)^n + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \tag{1.85}$$

$$= \sum_{n=0}^{\infty} (az)^n - 1 + \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \tag{1.86}$$

$$= \frac{1}{1-az} - 1 + \frac{1}{1-\frac{a}{z}} \tag{1.87}$$

where the expression for an infinite geometric progression has been used. Note, however, that each summation has different convergence constraints. Thus, note that the first summation only exists for $|az| < 1$, while the second summation only exists for $\left|\frac{a}{z}\right| < 1$. This means that the ROC for this transform is the ring $|a| < z < \frac{1}{|a|}$. The ROC is thus shown in Figure 1.18.

Combining the various terms and a slight rearrangement gives the expression

$$X(z) = \frac{1-a^2}{(1-az)(1-az^{-1})} \tag{1.88}$$

□

which has a zero at $z = 0$ and poles at $z = a$ and $z = \frac{1}{a}$.

#### 1.5.2.2 Properties of the $z$-transform

The power of the $z$-transform is a consequence of some very important properties that the transform possesses. Some of these properties are listed below, as a re-cap. Note that the proof of many of these properties follows immediately from the definition of the property itself and the $z$-transform, and is left as an exercise for the reader. Alternatively, cheat and look in, for example, [Proakis:1996].

**Linearity** If $x_1[n] \overset{z}{\rightleftharpoons} X_1(z)$ and $x_2[n] \overset{z}{\rightleftharpoons} X_2(z)$, then by linearity

$$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{z}{\rightleftharpoons} X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) \tag{P:3.2.1}$$

for any constants $\alpha_1$ and $\alpha_2$. Obviously, this property can be generalised for an arbitrary number of signals, and therefore if $x_m[n] \overset{z}{\rightleftharpoons} X_m(z)$ for $m = \{1, \dots, M\}$

$$x[n] = \sum_{m=1}^{M} \alpha_m x_m[n] \overset{z}{\rightleftharpoons} X(z) = \sum_{m=1}^{M} \alpha_m X_m(z) \tag{1.89}$$

for any constants $\{\alpha_m\}_1^M$.

**Time shifting** If $x[n] \overset{z}{\rightleftharpoons} X(z)$ then:

$$x[n-k] \overset{z}{\rightleftharpoons} z^{-k} X(z) \tag{1.90}$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

**Scaling in the $z$-domain** If $x[n] \overset{z}{\rightleftharpoons} X(z)$ with ROC $r_1 < |z| < r_2$, then

$$a^n\, x[n] \overset{z}{\rightleftharpoons} X(a^{-1}\, z) \quad \text{ROC: } |a|r_1 < |z| < |a|r_2 \tag{P:3.2.9}$$

for any constant $a$.

**Time reversal** If $x[n] \overset{z}{\rightleftharpoons} X(z)$ with ROC $r_1 < |z| < r_2$, then

$$x[-n] \overset{z}{\rightleftharpoons} X(z^{-1}) \quad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2} \tag{P:3.2.12}$$

**Differentiation in the $z$-domain** If $x[n] \overset{z}{\rightleftharpoons} X(z)$ then

$$nx[n] \overset{z}{\rightleftharpoons} -z\frac{dX(z)}{dz} \tag{P:3.2.14}$$

PROOF. Since

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\; z^{-n} \tag{1.91}$$

then differentiating both sides gives:

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} [nx[n]]\; z^{-n} = -z^{-1}\mathcal{Z}[nx[n]] \tag{1.92}$$

□

Both transforms have the same ROC.

**Convolution** If $x_1[n] \overset{z}{\rightleftharpoons} X_1(z)$ and $x_2[n] \overset{z}{\rightleftharpoons} X_2(z)$, then

$$x[n] = x_1[n] * x_2[n] \overset{z}{\rightleftharpoons} X(z) = X_1(z)X_2(z) \tag{3.2.17}$$

The ROC of $X(z)$ is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

PROOF. The convolution of $x_1[n]$ and $x_2[n]$ is defined as:

$$x[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \tag{1.93}$$

The $z$-transform of $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\; z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]\right] z^{-n} \tag{1.94}$$

Upon changing the order of the summations, then:

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \underbrace{\left[\sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n}\right]}_{X_2(z)\, z^{-k}} = X_2(z) \underbrace{\sum_{k=-\infty}^{\infty} x_1[k]\; z^{-k}}_{X_1(z)} \tag{1.95}$$

□

giving the desired result.

**The Initial Value Theorem** If $x[n] = 0,\ n < 0$ is a **causal** sequence, then

$$x[0] = \lim_{z\to\infty} X(z) \tag{P:3.2.23}$$

PROOF. Since $x[n]$ is causal, then:

$$X(z) = x[0] + x[1]\ z^{-1} + x[2]\ z^{-2} + \cdots \tag{1.96}$$

□

Hence, as $z \to \infty$, $z^{-n} \to 0$ since $n > 0$, and thus the desired result is obtained.

#### 1.5.2.3 The Unilateral $z$-transform

The two-sided $z$-transform requires that the corresponding signals be specified for the entire time range $n \in \mathbb{Z}$. This requirement prevents its used for systems that are described by difference equations with nonzero initial conditions. Since the input is applied at a finite time, say $n_0$, both input and output signals are specified for $n \geq n_0$, but are not necessarily zero for $n < 0$. Thus the two sided $z$-transform cannot be used

The one-sided **unilateral $z$-transform** of a signal $x[n]$ is defined by:

$$X^+(z) \equiv \sum_{n=0}^{\infty} x[n]\ z^{-n} \tag{P:3.5.1}$$

This is usually denoted as:

$$x[n] \overset{z^+}{\rightleftharpoons} X^+(z) \quad \text{or} \quad X^+(z) = \mathcal{Z}^+[x[n]] \tag{1.97}$$

The unilateral $z$-transform differs from the bilateral transform in the lower limit of the summation, which is always zero, whether or not the signal $x[n]$ is zero for $n < 0$ (i.e., causal). Therefore, the unilateral $z$-transform contains no information about the signal $x[n]$ for negative values of time, and is therefore *unique* only for causal signals. The unilateral and bilateral $z$-transforms are, consequentially, identical for the signal $x[n]\ u[n]$ where $u[n]$ is the step function. Since $x[n]\ u[n]$ is causal, the ROC of its transform, and hence the ROC of $X^+(z)$, is always the exterior of a circle. Thus, when discussing the unilateral $z$-transform, it is not necessary to refer to their ROC - which perhaps explains why this is the more commonly discussed transform in undergraduate courses.

Almost all the properties for the bilateral $z$-transform carry over to the unilateral transform with the exception of the shifting property.

**Shifting property: Time Delay** If $x[n] \overset{z^+}{\rightleftharpoons} X^+(z)$ then:

$$x[n-k] \overset{z^+}{\rightleftharpoons} z^{-k} X(z) + \underbrace{\sum_{n=-k}^{-1} x[n]\ z^{-(n+k)}}_{\text{initial conditions}}, \quad k > 0 \tag{1.98}$$

PROOF. Since

$$X^+(z) \equiv \sum_{n=0}^{\infty} x[n]\ z^{-n} \tag{P:3.5.1}$$

then it follows that

$$\mathcal{Z}^+[x[n-k]] = \sum_{n=0}^{\infty} x[n-k]\; z^{-n} = \sum_{m=-k}^{\infty} x[m]\; z^{-(m+k)} \tag{1.99}$$

by the change of index $m = n - k$,

$$= z^{-k} \sum_{m=-k}^{-1} x[m]\; z^{-m} + z^{-k} \underbrace{\sum_{m=0}^{\infty} x[m]\; z^{-m}}_{X^+(z)} \tag{1.100}$$

□

This is the desired result.

**Shifting property: Time Advance** If $x[n] \overset{z^+}{\rightleftharpoons} X^+(z)$ then:

$$x[n+k] \overset{z^+}{\rightleftharpoons} z^k X(z) - \sum_{n=0}^{k-1} x[n]\; z^{k-n}, \quad k > 0 \tag{1.101}$$

PROOF. From the definition of the unilateral transform, it follows

$$\mathcal{Z}^+[x[n+k]] = \sum_{n=0}^{\infty} x[n+k]\; z^{-n} = \sum_{m=k}^{\infty} x[m]\; z^{-(m-k)} \tag{1.102}$$

by the change of index $m = n + k$. Thus,

$$= z^k \underbrace{\sum_{0}^{\infty} x[m]\; z^{-m}}_{X^+(z)} - z^k \sum_{m=1}^{k-1} x[m]\; z^{-m} \tag{1.103}$$

□

This is the desired result.

**Final Value Theorem** If $x[n] \overset{z^+}{\rightleftharpoons} X^+(z)$ then:

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1) X^+(z) \tag{P:3.5.6}$$

The limit on the right hand side (RHS) exists if the ROC of $(z-1)X^+(z)$ includes the unit circle.

Further information can be found in books on discrete-time systems, for example [Proakis:1996, Section 3.5, Page 197].

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### 1.5.3 Review of linear time-invariant systems

- Systems which are **LTI** can be elegantly analysed in both the time and frequency domain: **convolution** in time, multiplication in frequency.
- For signals and sequences, it is common to write $\{y[n]\}_{n=-\infty}^{\infty}$, or even $\{y[n]\}_{n\in\mathbb{Z}}$ rather than simply $y[n]$: the latter is sufficient for these notes.

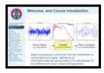
- Output, $y[n]$, of a **LTI** system is the **convolution** of the input, $x[n]$, and the **impulse response** of the system, $h[n]$:

$$y[n] = x[n] * h[n] \triangleq \sum_{k \in \mathbb{Z}} x[k]\; h[n-k] \tag{M:2.3.2}$$

- By making the substitution $\hat{k} = n - k$, it follows:

$$y[n] = \sum_{k \in \mathbb{Z}} h[k]\; x[n-k] = h[n] * x[n] \tag{M:2.3.3}$$

### 1.5.3.1 Matrix-vector formulation for convolution



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If $x[n]$ and $h[n]$ are sequences of finite duration, the **convolution** operation can be written in matrix-vector form. Let $x[n]\,,\, 0 \le n \le N-1$ and $h[n]\,,\, 0 \le n \le M-1$ be finite-duration sequences, then $y[n]\,,\, 0 \le n \le L-1$, where $L = N + M - 1$, can be written as:

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[M-1] \\ \vdots \\ y[N-1] \\ \vdots \\ y[L-2] \\ y[L-1] \end{bmatrix} = \begin{bmatrix} x[0] & 0 & \cdots & 0 \\ x[1] & x[0] & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x[M-1] & \cdots & \cdots & x[0] \\ \vdots & \ddots & \ddots & \vdots \\ x[N-1] & \cdots & \cdots & x[N-M] \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & x[N-1] & x[N-2] \\ 0 & \cdots & 0 & x[N-1] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[M-1] \end{bmatrix} \tag{M:2.3.4}$$

or

$$\mathbf{y} = \mathbf{X}\,\mathbf{h} \tag{M:2.3.5}$$

- Here, $\mathbf{y} \in \mathbb{R}^L$, $\mathbf{X} \in \mathbb{R}^{L \times M}$, and $\mathbf{h} \in \mathbb{R}^M$.
- The matrix $\mathbf{X}$ is termed an **input data matrix**, and has the property that it is **toeplitz**. [1]
- The observation or output vector $\mathbf{y}$ can also be written in a similar way as:

$$\mathbf{y} = \mathbf{H}\,\mathbf{x} \tag{M:2.3.6}$$

  in which $\mathbf{H}$ is also **toeplitz**.
- A system is **causal** if the present output sample depends only on past and/or present input samples.
- Assume system is asymptotically stable.

### 1.5.3.2 Transform-domain analysis of LTI systems

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Time-domain **convolution**:

$$y[n] = \sum_{k \in \mathbb{Z}} x[k]\; h[n-k] \tag{M:2.3.2}$$

[1] A **Toeplitz** matrix is one in which the elements along each diagonal, parallel to the main diagonal each descending from left to right, are constant. Note that the anti-diagonals are not necessarily equal.

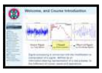
or

$$y[n] = \sum_{k \in \mathbb{Z}} h[k]\; x[n-k] \tag{M:2.3.3}$$

Taking $z$-**transforms** gives:

$$Y\left(z\right) = H\left(z\right)\; X\left(z\right) \tag{M:2.3.8}$$

where $X\left(z\right)$, $Y\left(z\right)$ and $H\left(z\right)$ are the $z$-**transforms** of the input, output, and impulse response sequences respectively. $H\left(z\right) = \mathcal{Z}[h[n]]$ is the **system function** or **transfer function**.



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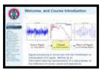
#### 1.5.3.3 Frequency response of LTI systems

The **frequency response** of the system is found by evaluating the $z$-transform on the unit circle, so $z = e^{j\omega}$:

$$Y\left(e^{j\omega T}\right) = H\left(e^{j\omega T}\right)\; X\left(e^{j\omega T}\right) \tag{M:2.3.9}$$

- $|H(e^{j\omega})|$ is the **magnitude response** of the system, and $\arg H(e^{j\omega})$ is the **phase response**.
- The **group delay** of the system is a measure of the average delay of the system as a function of frequency:

$$\tau(e^{j\omega}) = -\frac{d}{d\omega} \arg H(e^{j\omega}) \tag{M:2.3.11}$$



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#### 1.5.3.4 Frequency response to Periodic Inputs

Although the convolution summation formula can be used to compute the response of a stable system to any input, the frequency-domain input-output relationship for a **LTI** cannot be used with periodic inputs, since periodic signals do not strictly possess a $z$-transform. However, it is possible to develop an expression for the frequency response of LTI from first principles. Let $x[n]$ be a periodic signal with fundamental period $N$. This signal can be expanded using an IDFT as:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}kn}, \quad n \in \{0, \dots, N-1\} \tag{M:2.3.19}$$

where $X_k$ are the Fourier components.

Hence, it follows that on substitution into the convolution equation:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]\; x[n-m] = \frac{1}{N} \sum_{m=-\infty}^{\infty} h[m] \sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}k(n-m)} \tag{M:2.3.20}$$

which, by interchanging the order of summation (noting that the limits are over a rectangular region of summation), gives;

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k\, e^{j\frac{2\pi}{N}kn} \underbrace{\sum_{m=-\infty}^{\infty} h[m]\; e^{-j\frac{2\pi}{N}km}}_{H(e^{j\frac{2\pi}{N}k})} \tag{1.104}$$

where $H(e^{j\frac{2\pi}{N}k})$ are samples of $H(e^{j\omega})$. Hence,

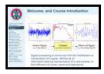
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ H(e^{j\frac{2\pi}{N}k})\; X_k \right\} e^{j\frac{2\pi}{N}kn} \tag{1.105}$$

However, this is just the inverse-DFT expansion of $y[n]$, and therefore:

$$Y_k = H(e^{j\frac{2\pi}{N}k})\,X_k \quad k \in \{0,\dots,N-1\} \tag{M:2.3.21}$$

Thus, the response of a LTI system to a periodic input is also periodic with the same period. The magnitude of the input components is modified by $|H(e^{j\frac{2\pi}{N}k})|$, and the phase is modified by $\arg H(e^{j\frac{2\pi}{N}k})$.

### 1.5.4 Rational transfer functions



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Many systems can be expressed in the $z$-domain by a **rational transfer function**. They are described in the time domain by:

$$y[n] = -\sum_{k=1}^{P} a_k\, y[n-k] + \sum_{k=0}^{Q} d_k\, x[n-k] \tag{M:2.3.12}$$

Taking $z$-transforms gives:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{Q} d_k\, z^{-k}}{1 + \sum_{k=1}^{P} a_k\, z^{-k}} \triangleq \frac{D(z)}{A(z)} \tag{M:2.3.13}$$

This can be described in the complex $z$-plane as:

$$H(z) = \frac{D(z)}{A(z)} = G\frac{\prod_{k=1}^{Q}(1 - z_k\, z^{-1})}{\prod_{k=1}^{P}(1 - p_k\, z^{-1})} \tag{M:2.3.14}$$

where $p_k$ are the poles of the system, and $z_k$ are the zeros.





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# 2

# Discrete-Time Stochastic Processes

Introduces the notion of time-series or random processes. Gives an interpretation using ensembles, and covers second-order statistics including correlation sequences. Discusses types of stationary processes, ergodicity, joint-signal statistics, and correlation matrices.

## 2.1 A Note on Notation

Note that, unfortunately, for this module, a slightly different (and abusive use of) notation for random quantities is used than what was presented in the first four handouts of the *Probability, Random Variables, and Estimation Theory (PET)* module. In the literature, most time series are described using lower-case letters, primarily since once the notation for the representation of a random process in the frequency domain is discussed, upper-case letters are exclusively reserved to denote spectral representations. Moreover, lower-case letters for time-series are generally more recognisable and readable, and helps with the clarity of the presentation. Hence, random variables and vectors in this handout will not always be denoted using upper-case letters.

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## 2.2 Definition of a Stochastic Process

After studying random variables and vectors, these concepts can now (easily) be extended to discrete-time signals or sequences.

- Natural discrete-time signals can be characterised as random signals, since their values cannot be determined precisely; that is, they are **unpredictable**. A natural mathematical framework for the description of these discrete-time random signals is provided by discrete-time stochastic processes.

- To obtain a formal definition, consider an experiment with a finite or infinite number of unpredictable outcomes from a sample space $\mathcal{S} = \{\zeta_k,\, k \in \mathbb{Z}^+\}$, each occurring with probability $\Pr(\zeta_k)$. Assign by some rule to each $\zeta_k \in \mathcal{S}$ a deterministic sequence $x[n, \zeta_k]\,,\ n \in \mathbb{Z}$.

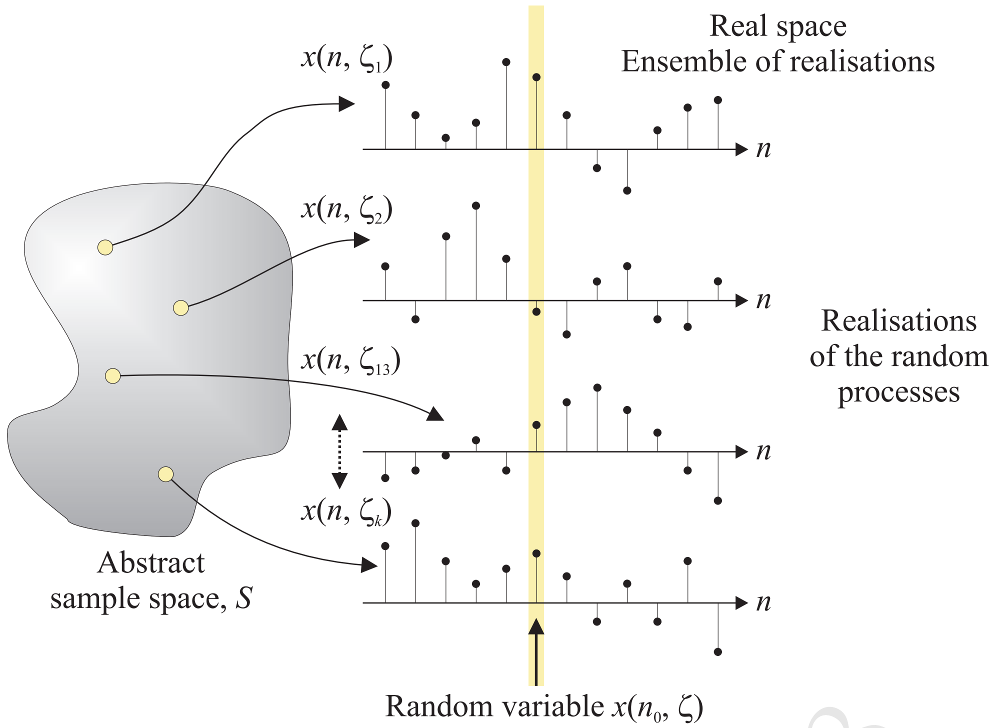
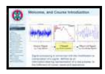



Figure 2.1: A graphical representation of a random process.

- The sample space $\mathcal{S}$, probabilities $\Pr(\zeta_k)$, and the sequences $x[n, \zeta_k]$, $n \in \mathbb{Z}$ constitute a **discrete-time stochastic process**, or **random sequence**.
- Formally, $x[n, \zeta_k]$, $n \in \mathbb{Z}$ is a random sequence or **stochastic process** if, for a fixed value $n_0 \in \mathbb{Z}^+$ of $n$, $x[n_0, \zeta]$, $n \in \mathbb{Z}$ is a random variable.
- A random or stochastic process is also known as a **time series** in the statistics literature.
- It is an infinite sequence of random variables, so could be thought of as an infinite-dimensional random vector. Indeed, finite-length random signals and sequences can specifically be represented by the concept of a random vector.

### 2.2.1 Interpretation of Sequences



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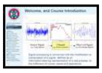
The set of all possible sequences $\{x[n, \zeta]\}$ is called an **ensemble**, and each individual sequence $x[n,\ \zeta_k]$, corresponding to a specific value of $\zeta = \zeta_k$, is called a **realisation** or a **sample sequence** of the ensemble. Hence, when a random process is observed through the outcome of a single experiment, one member of the ensemble is selected randomly and presented. A graphical representation of a random process is shown in Figure 2.1.

There are four possible interpretations of $x[n, \zeta]$:

| | $\zeta$ Fixed | $\zeta$ Variable |
|---|---|---|
| $n$ Fixed | Number | Random variable |
| $n$ Variable | Sample sequence | Stochastic process |

Use simplified notation $x[n] \equiv x[n, \zeta]$ to denote both a stochastic process, and a single realisation. The word *stochastic* is derived from the Greek word *stochasticos*, which means skillful in aiming or guessing. Use the terms **random process** and **stochastic process** interchangeably throughout this course.



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### 2.2.2 Predictable Processes

A deterministic signal is by definition exactly predictable; it assumes there exists a certain functional relationship that completely describes the signal, even if that functional relationship is not available or is extremely difficult to describe. The unpredictability of a random process is, in general, the combined result of the following two characteristics:

1. The selection of a single realisation of a stochastic process is based on the outcome of a random experiment; in other-words, it depends on $\zeta$.

2. No functional description is available for *all* realisations of the *ensemble*. In other-words, even if a functional relationship is available for a subset of the ensemble, it might not be available for all members of the ensemble.

In some special cases, however, a functional relationship is available. This means that after the occurrence of all samples of a particular realisation up to a particular point, $n$, all future values can be predicted exactly from the past ones.

If this is the case for a random process, then it is called **predictable**, otherwise it is said to be **unpredictable** or a **regular process**.

**KEYPOINT! (Predictable Process).** As an example of a predicatble process, consider the signal:

$$x[n,\zeta] = A\,\sin\left(\omega\, n + \phi\right) \tag{2.1}$$

□

where $A$ is a known amplitude, $\omega$ is a known normalised angular frequency, and $\phi$ is a random phase, where $\phi \sim f_\Phi\left(\phi\right)$ is its probability density function (pdf).

As an outline of this idea, suppose that all the samples of a stochastic process $x(n,\zeta)$ upto sample $n-1$ are known; thus, $\{x[k,\zeta]\}_{k=-\infty}^{n-1}$ are known. Then the predicted value of $x[n]$ might, for example, be expressed as:

$$\hat{x}[n] = -\sum_{k=1}^{\infty} a_k^*\, x[n-k] \tag{T:7.189}$$

The error in this prediction is given by

$$\epsilon[n] = x[n] - \hat{x}[n] = \sum_{k=0}^{\infty} a_k^*\, x[n-k] \tag{T:7.190}$$

where $a_0 = 1$. The process is said to be **predictable** if the $\{a_k\}$'s can be chosen such that:

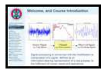
$$\sigma_\epsilon^2 = \mathbb{E}\left[|\epsilon[n]|^2\right] = 0 \tag{T:7.191}$$

Otherwise the process is not predictable. The phrase *not predictable* is somewhat misleading, since the **linear prediction** in Equation T:7.189 can be applied to any process, whether predictable or not, with satisfactory results. If a process is not predictable, it just means that the prediction error variance is not zero.

An example of **predictable process** is the process $x[n,\zeta] = c$, where $c$ is a random variable, since every realisation of the discrete-time signal has a constant amplitude, and once $x[n_0,\zeta_k]$ is known for a particular realisation, all other samples of that process have also been determined.

The notion of predictable and regular processes is formally presented through the **Wold decomposition**, and further details of this very important theorem can be found in [Therrien:1992, Section 7.6, Page 390] and [Papoulis:1991, Page 420].

### 2.2.3 Description using pdfs



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For fixed $n = n_0$, it is clear from Figure 2.1 that $x[n_0, \zeta]$ is a random variable. Moreover, the random vector formed from the $k$ random variables $\{x[n_j], j \in \{1, \dots k\}\}$ is characterised by the joint-cumulative distribution function (cdf) and pdfs:

$$F_X(x_1 \dots x_k \mid n_1 \dots n_k) = \Pr(x[n_1] \le x_1, \dots, x[n_k] \le x_k) \tag{2.2}$$

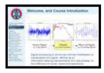
$$f_X(x_1 \dots x_k \mid n_1 \dots n_k) = \frac{\partial^k F_X(x_1 \dots x_k \mid n_1 \dots n_k)}{\partial x_1 \cdots \partial x_k} \tag{2.3}$$

In exactly the same way as with random variables and random vectors, it is:

- difficult to estimate these probability functions without considerable additional information or assumptions;
- possible to frequently characterise stochastic processes usefully with much less information.

Thus, the density and distribution functions are characterised using moments and, in particular, second-order moments.

## 2.3 Second-order Statistical Description



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Random variables can be characterised, upto second-order statistics, using the mean and variance; random vectors are characterised by the mean vector, auto-correlation and auto-covariance matrices. Random processes, however, are characterised by sequences, where a particular sample, $n_0$, of this sequence characterises the random variable $x[n_0, \zeta]$. These sequences are the mean and variance sequence, the autocorrelation and autocovariance sequences, as outlined below.

**Mean and Variance Sequence** At time $n$, the **ensemble** mean and variance are given by:

$$\mu_x[n] = \mathbb{E}[x[n]] \tag{M:3.3.3}$$

$$\sigma_x^2[n] = \mathbb{E}\left[|x[n] - \mu_x[n]|^2\right] = \mathbb{E}\left[|x[n]|^2\right] - |\mu_x[n]|^2 \tag{M:3.3.4}$$

Both $\mu_x[n]$ and $\sigma_x^2[n]$ are deterministic sequences.

**Autocorrelation sequence** The second-order statistic $r_{xx}[n_1, n_2]$ provides a measure of the dependence between values of the process at two different times; it can provide information about the time variation of the process:

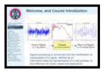
$$r_{xx}[n_1, n_2] = \mathbb{E}[x[n_1]\; x^*[n_2]] \tag{M:3.3.5}$$

**Autocovariance sequence** The autocovariance sequence provides a measure of how similar the deviation from the mean of a process is at two different time instances:

$$\begin{aligned}\gamma_{xx}[n_1, n_2] &= \mathbb{E}[(x[n_1] - \mu_x[n_1])(x[n_2] - \mu_x[n_2])^*] \\ &= r_{xx}[n_1, n_2] - \mu_x[n_1]\; \mu_x^*[n_2]\end{aligned} \tag{M:3.3.6}$$

To show how these deterministic sequences of a stochastic process can be calculated, several examples are considered in detail below.

### 2.3.1 Example of calculating autocorrelations



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These examples assume that the notion of stationarity has been met; this, in fact, is not discussed until Section 2.5. Either the reader can skip these examples and return to read them after reading Section 2.5, or for the moment the reader can proceed by using the simple definition that a "stationary" process is one for which the autocorrelation function $r_{xx}(n,m) = r_{xx}(n-m) = r_{xx}(l)$ is simply a function of the time (or sample index) differences, also called the lag: $l = n - m$.

**Example 2.1 (Straightforward example).** A random variable $y(n)$ is defined to be:

$$y(n) = x(n) + x(n+m) \tag{2.4}$$

where $m$ is some integer, and $x(n)$ is a stationary stochastic process whose autocorrelation function is given by:

$$r_{xx}(l) = e^{-l^2} \tag{2.5}$$

Derive an expression for the autocorrelation of the stochastic process $y(n)$.

SOLUTION. In this example, it is simplest to form the product:

$$\begin{aligned} y(n)\, y^*(n-l) &= [x(n) + x(n+m)]\; [x^*(n-l) + x^*(n+m-l)] && (2.6) \\ &= x(n)\, x^*(n-l) + x(n+m)\, x^*(n-l) \\ &\quad + x(n)\, x^*(n+m-l) + x(n+m)\, x^*(n+m-l) && (2.7) \end{aligned}$$

then, taking expectations, noting $x(n)$ is a stationary signal, it follows:

$$r_{yy}(l) = r_{xx}(l) + r_{xx}(m+l) + r_{xx}(l-m) + r_{xx}(l) \tag{2.8}$$

giving, in this particular case,

$$r_{yy}(l) = 2\, e^{-l^2} + e^{-(l+m)^2} + e^{-(l-m)^2} \tag{2.9}$$

□

**Example 2.2 ( [Manolakis:2000, Ex 3.9, page 144]).** The harmonic process $x[n]$ is defined by:

$$x[n] = \sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k), \quad \omega_k \neq 0 \tag{M:3.3.50}$$

where $M$, $\{A_k\}_1^M$ and $\{\omega_k\}_1^M$ are constants, and $\{\phi_k\}_1^M$ are pairwise independent random variables uniformly distributed in the interval $[0, 2\pi]$.

1. Determine the mean of $x(n)$.

2. Show the autocorrelation sequence is given by

$$r_{xx}[\ell] = \frac{1}{2} \sum_{k=1}^{M} |A_k|^2 \cos \omega_k \ell, \quad -\infty < \ell < \infty \tag{2.10}$$

SOLUTION. 1. The expected value of the process is straightforwardly given by:

$$\mathbb{E}[x(n)] = \mathbb{E}\left[\sum_{k=1}^{M} A_k \cos(\omega_k n + \phi_k)\right] = \sum_{k=1}^{M} A_k \,\mathbb{E}[\cos(\omega_k n + \phi_k)] \tag{2.11}$$

Recall from results derived earlier in the course that if $x(n,\zeta) = g(n, \phi(\zeta))$ is a random variable obtained by transforming $\phi(\zeta)$ through a known function, $g$, the expectation of $x(n) = x(n,\zeta)$ is:

$$\mathbb{E}[x(n)] = \int_{-\infty}^{\infty} g(n,\phi)\, p_\Phi(\phi)\, d\phi \tag{2.12}$$

It is important to consider $n$ as a constant.

Since a co-sinusoid is zero-mean, then:

$$\mathbb{E}[\cos(\omega_k n + \phi_k)] = \int_0^{2\pi} \cos(\omega_k n + \phi_k) \times \frac{1}{2\pi} \times d\phi_k = 0 \tag{2.13}$$

Hence, it follows:

$$\mathbb{E}[x(n)] = 0, \quad \forall n \tag{2.14}$$

2. The autocorrelation $r_{xx}(n_1, n_2) = \mathbb{E}[x(n_1)\, x^*(n_2)]$ follows similarly:

$$r_{xx}(n_1,n_2) = \mathbb{E}\left[\sum_{k=1}^{M} A_k \cos(\omega_k n_1 + \phi_k) \sum_{j=1}^{M} A_j^* \cos(\omega_j n_2 + \phi_j)\right] \tag{2.15}$$

$$= \sum_{k=1}^{M}\sum_{j=1}^{M} A_k\, A_j^* \mathbb{E}[\cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)] \tag{2.16}$$

After some algebra, it can be shown that:

$$\mathbb{E}[\cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)] = \begin{cases} \frac{1}{2}\cos\omega_k(n_1 - n_2) & k = j \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

The proof of this statement is obtained by considering the term

$$r(\phi_k, \phi_j) = \mathbb{E}[\cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)] \tag{2.18}$$

for the cases when $k \neq j$, and when $k = j$. Considering the former case first, $k \neq j$, then

$$r(\phi_k, \phi_j) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)\, d\phi_j\, d\phi_k \tag{2.19}$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi} \cos(\omega_k n_1 + \phi_k)\, d\phi_k \int_0^{2\pi} \cos(\omega_j n_2 + \phi_j)\, d\phi_j \tag{2.20}$$

$$= 0 \tag{2.21}$$

An alternative derivation which might be considered more straightforward is to observe that Equation 2.18 might also be written as:

$$r(\phi_k, \phi_j) = \mathbb{E}[g(\phi_k)\, h(\phi_j)] = \mathbb{E}[g(\phi_k)]\, \mathbb{E}[h(\phi_j)] \tag{2.22}$$

where $g(\phi_k) = \cos(\omega_k n_1 + \phi_k)$ and $h(\phi_k) = \cos(\omega_j n_2 + \phi_j)$, and the fact that $\phi_k$ and $\phi_j$ are independent implies the expectation function may be factorised.

For the case when $k = j$ such that $\phi = \phi_k = \phi_j$ and $\omega = \omega_k = \omega_j$, then:

$$r(\phi, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega n_1 + \phi)\, \cos(\omega n_2 + \phi)\, d\phi \tag{2.23}$$

Using the trigonometric identity $\cos A\, \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$, then:

$$\begin{aligned} r(\phi_k, \phi_j) &= \frac{1}{4\pi} \int_0^{2\pi} \left\{\cos \omega (n_1 - n_2) + \cos(\omega(n_1 + n_2) + 2\phi)\right\} d\phi & (2.24)\\ &= \frac{1}{2} \cos \omega (n_1 - n_2) & (2.25)\end{aligned}$$

giving the result above; namely:

$$\mathbb{E}\left[\cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)\right] = \frac{1}{2} \cos \omega_k (n_1 - n_2)\, \delta(k - j) \tag{2.26}$$

Substituting this expression into

$$r_{xx}(n_1, n_2) = \frac{1}{2} \sum_{k=1}^{M} \sum_{j=1}^{M} A_k\, A_j^* \mathbb{E}\left[\cos(\omega_k n_1 + \phi_k)\, \cos(\omega_j n_2 + \phi_j)\right] \tag{2.27}$$

thus leads to the desired result. It can be seen that the process $x(n)$ must be a stationary process, as it is only a function of the lag $l$:

$$r_{xx}(l) = \frac{1}{2} \sum_{k=1}^{M} |A_k|^2\, \cos \omega_k l, \quad -\infty < l < \infty \tag{2.28}$$

□

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## 2.4 Types of Stochastic Processes

Some useful types of stochastic properties, based on their statistical properties, are now introduced:

**Independence** A stochastic process is independent if, and only if, (iff)

$$f_X\left(x_1, \dots, x_N \mid n_1, \dots, n_N\right) = \prod_{k=1}^{N} f_{X_k}\left(x_k \mid n_k\right) \tag{M:3.3.10}$$

$\forall N,\, n_k,\, k \in \{1, \dots, N\}$. Here, therefore, $x(n)$ is a sequence of independent random variables.

**An independent and identically distributed (i. i. d.) process** is one where all the random variables $\{x(n_k, \zeta),\, n_k \in \mathbb{Z}\}$ have the same pdf, and $x(n)$ will be called an **i. i. d.** random process.

**Example 2.3 (Independence: i. i. d. processes).** I am selling my house, and have decided to accept the first offer exceeding $K$ pounds. Assuming that the offers are i. i. d. random variables, with common cumulative distribution function $F_X(x)$, where $x$ is the offer price, find the expected number of offers received before I sell the house.

**This is only the first 50 pages of a 192 page handout.**

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