Convex Optimization

Fundamentals and Applications in Statistical Signal Processing

João Mota

EURASIP/UDRC Summer School 2019

Heriot-Watt University

Optimization Problems

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- $x \in \mathbb{R}^n$: optimization variable
- $f : \mathbb{R}^n \to \mathbb{R}$: cost function (or objective)
- $\Omega \subset \mathbb{R}^n$: constraint set

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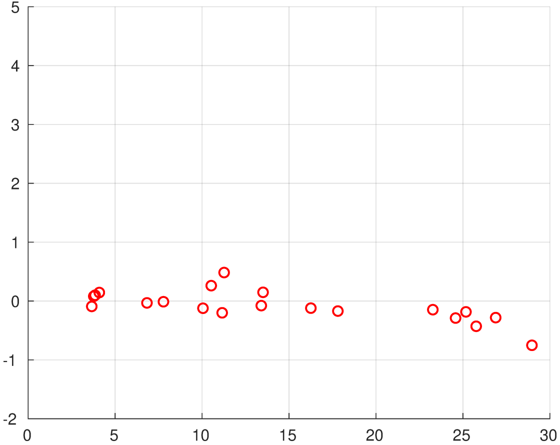

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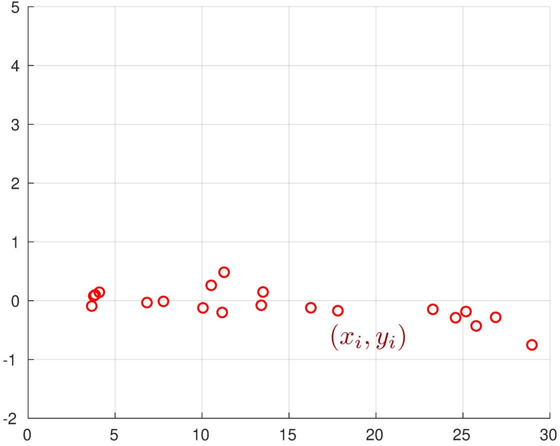

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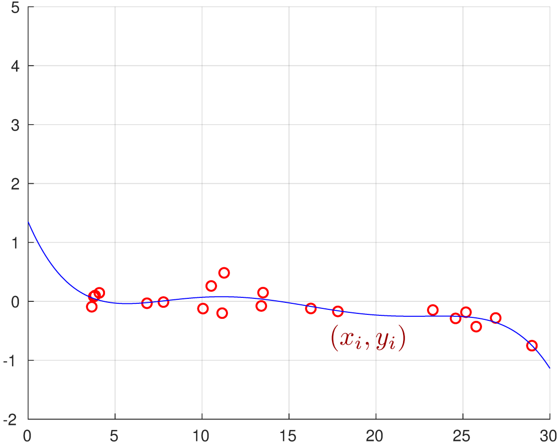

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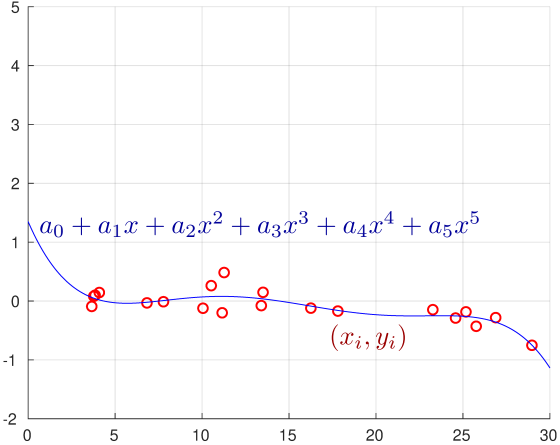

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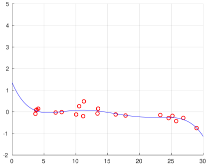

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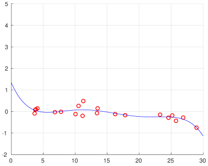

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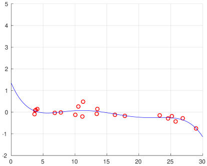

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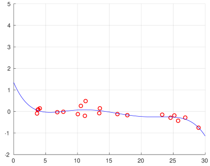

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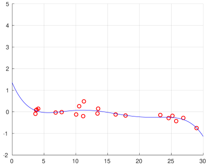

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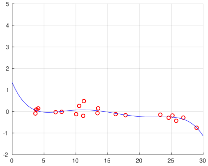

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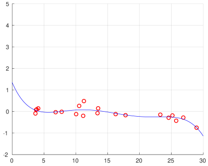

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What if there are outliers?

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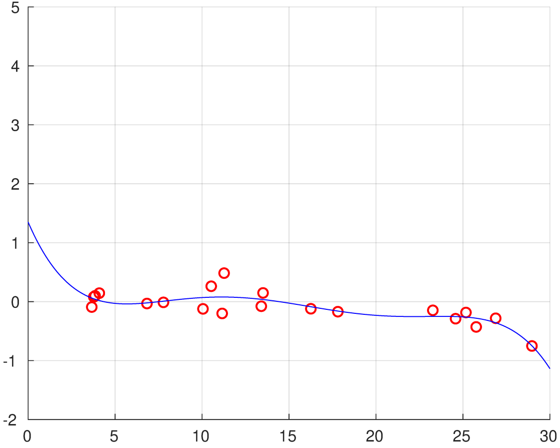

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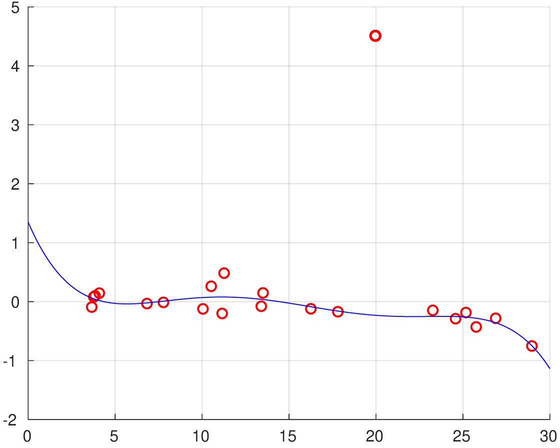

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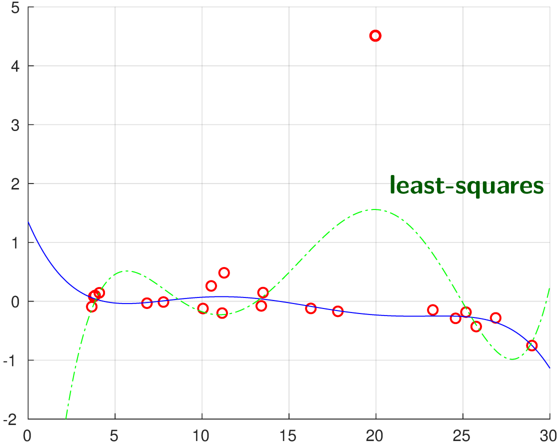

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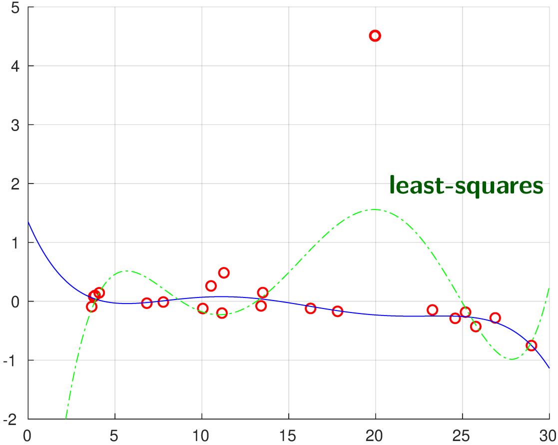

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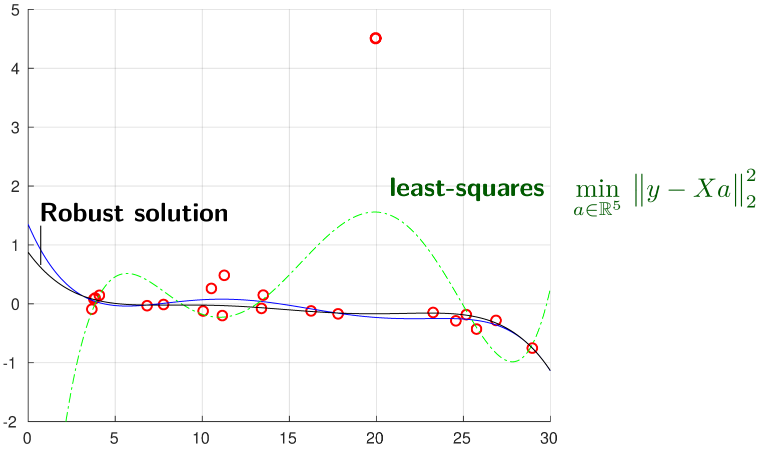

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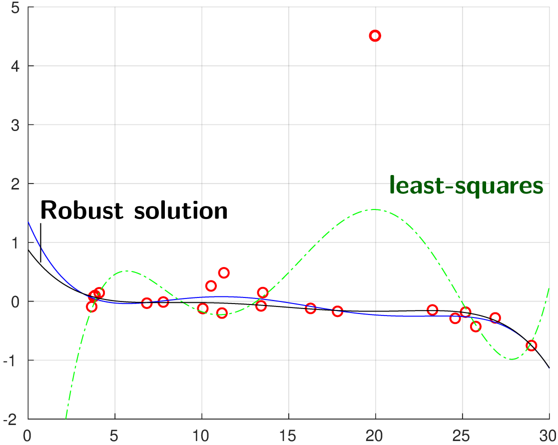

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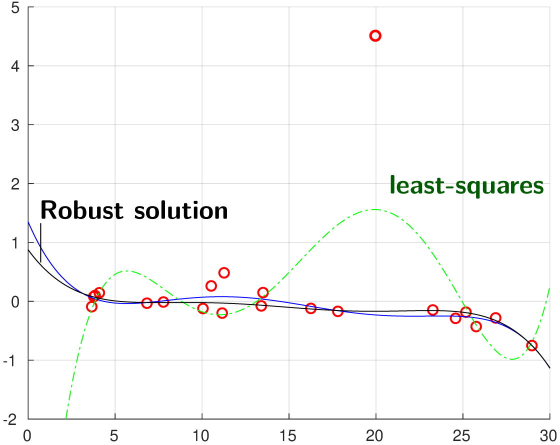

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no closed-form

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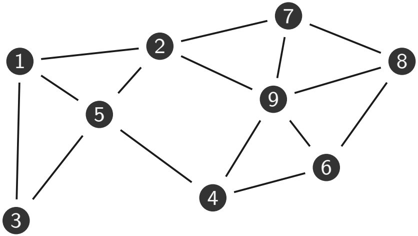

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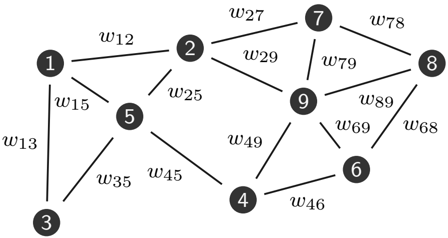

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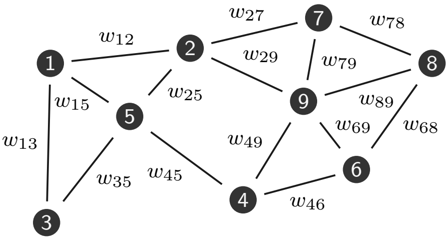

Cut: set of edges whose removal splits the graph into two

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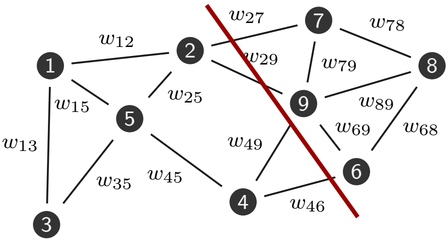

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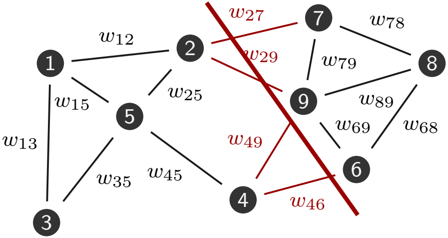

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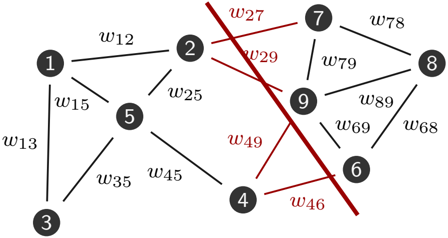

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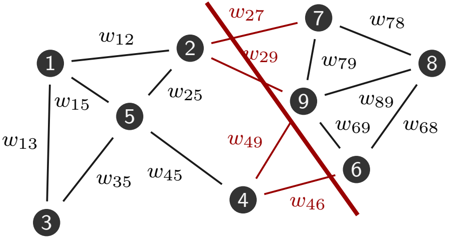

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Difficulty of optimization problems

- Closed-form solution ***(easy)***

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- Combinatorial, NP-Hard, requires exhaustive search ***(hard)***

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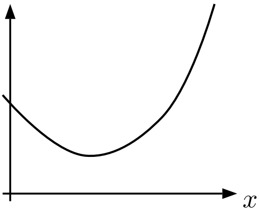
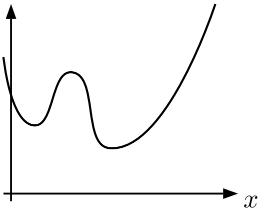
convex



nonconvex



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- Lots of *applications:* machine learning, communications, economics and finance, control systems, electronic circuit design, statistics, etc.
- Many algorithms for *nonconvex optimization* use convex surrogates

Convex problems

Hierarchical classification (specialized solvers):

Convex problems

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LP

linear programming

Convex problems

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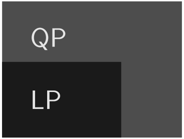

quadratic programming

linear programming

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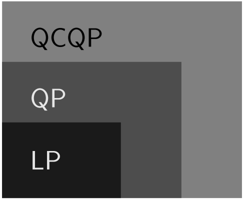

quadratically constrained QP

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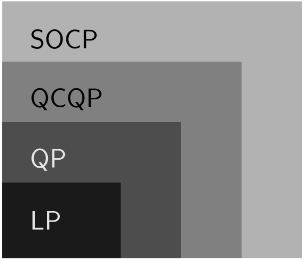

second-order cone programming

quadratically constrained QP

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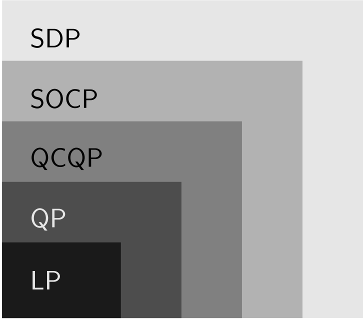

semidefinite programming

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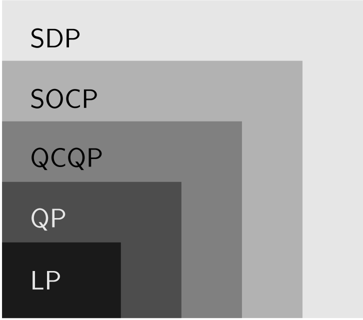

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linear programming

Other classifications:

differentiable vs. *nondifferentiable* programming

unconstrained vs. *constrained* programming

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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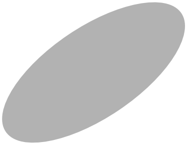

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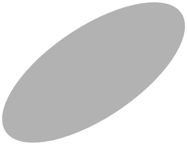

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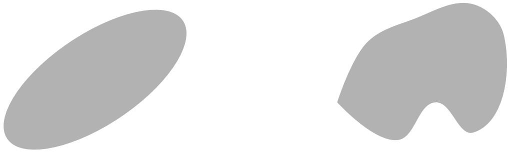

convex **nonconvex**

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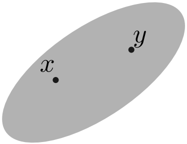

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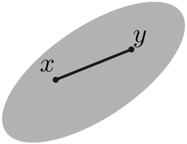

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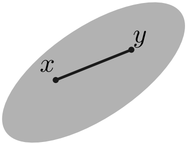

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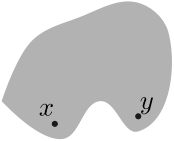

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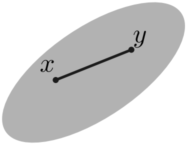

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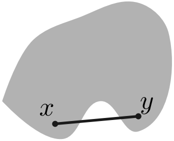

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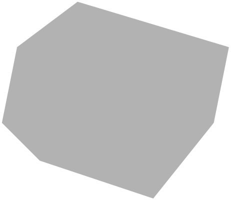

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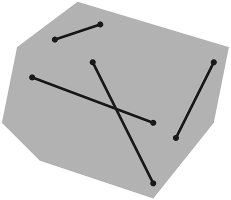

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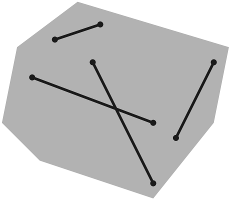

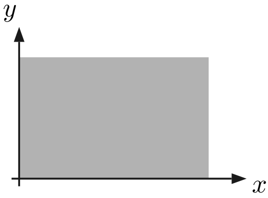

Examples of nonconvex sets

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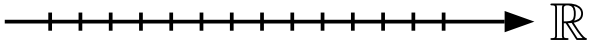
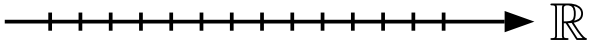


discrete sets

Examples of nonconvex sets



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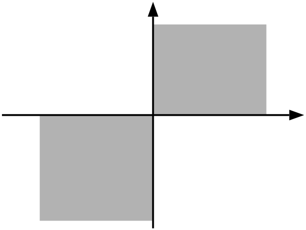

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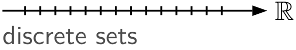

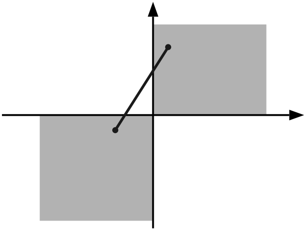

How to identify convex sets?

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vocabulary $+$ **grammar**

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simple sets

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vocabulary + **grammar**

simple sets — *operations preserving convexity*

Simple sets

Simple sets

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$$\mathcal{H}_{a,b} = \left\{ x \in \mathbb{R}^n \,:\, a^\top x = b \right\}$$

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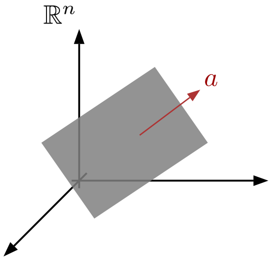

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$\mathbb{R}^2$

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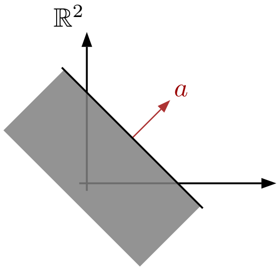

Simple sets

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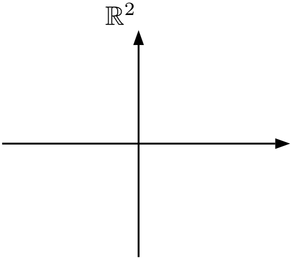

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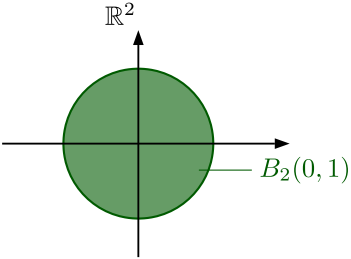

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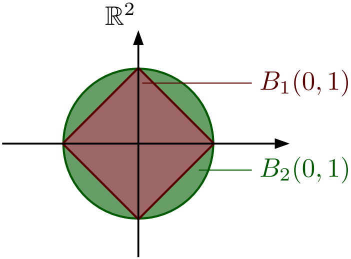

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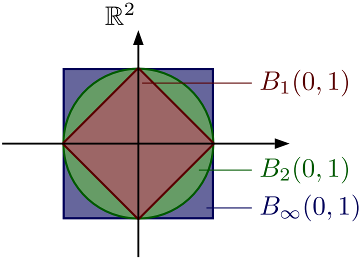

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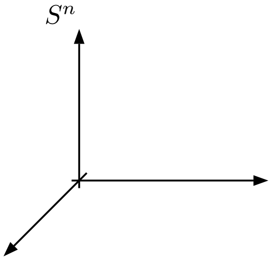

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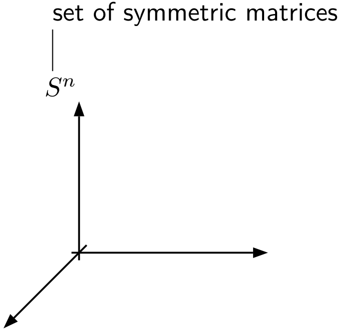

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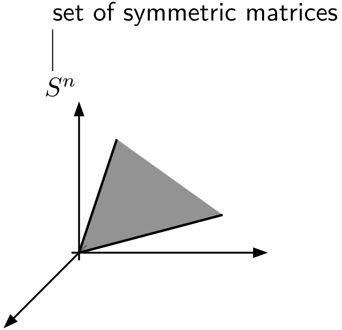

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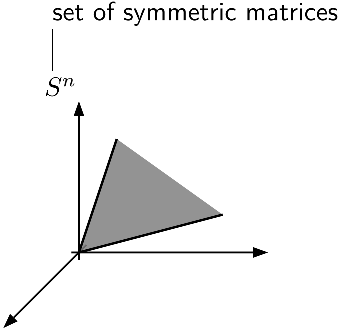

$$X \succeq 0_{n\times n} \qquad \Longleftrightarrow \qquad \lambda_{\min}(X) \geq 0 \qquad \Longleftrightarrow \qquad v^\top X v \geq 0\,, \quad \forall v$$

How to identify convex sets?

vocabulary	$+$	**grammar**
simple sets		*operations preserving convexity*

How to identify convex sets?

vocabulary + **grammar**

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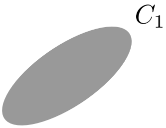

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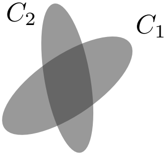

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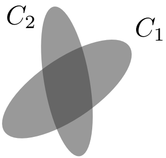

Intersection

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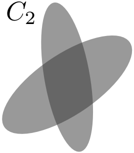

Intersection

$C_1, C_2, \ldots, C_m$: convex

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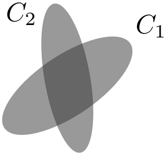

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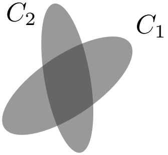

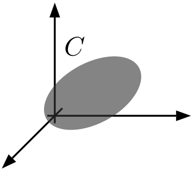

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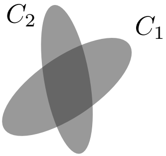

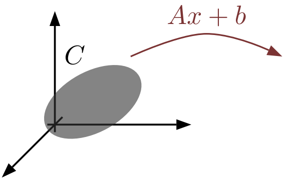

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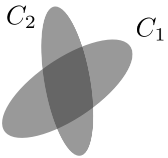

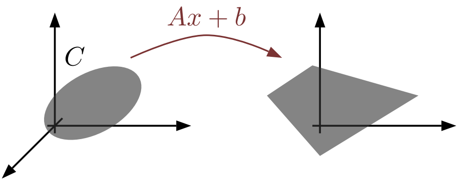

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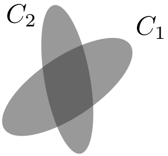

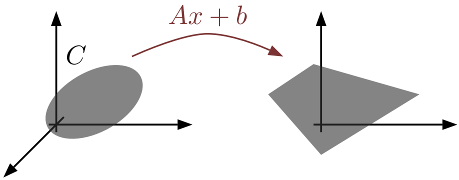

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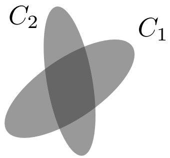
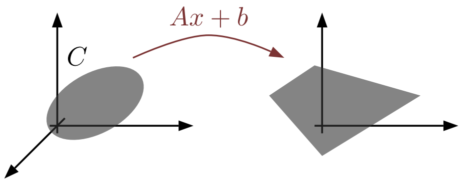
Affine operations

How to identify convex sets?





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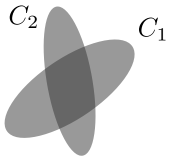

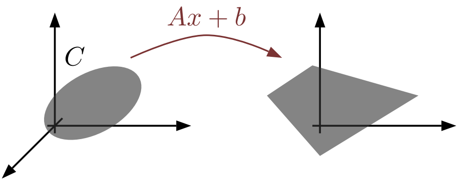

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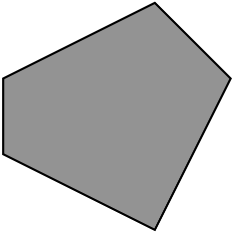

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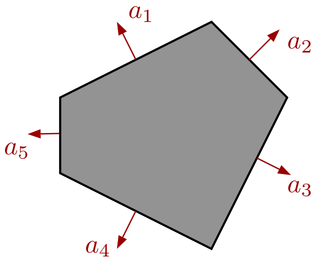

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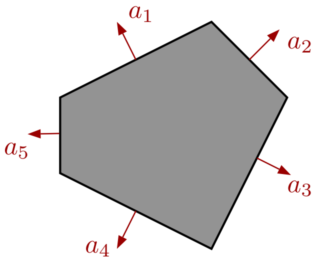

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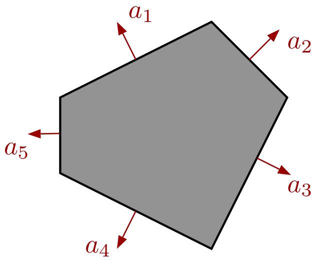

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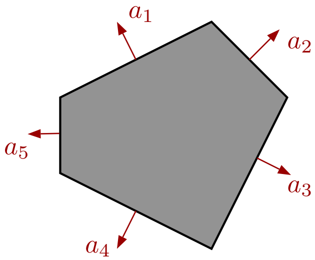

Example

Ellipsoids

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Ellipsoids $\big(A \succ 0\big)$

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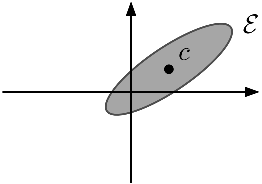

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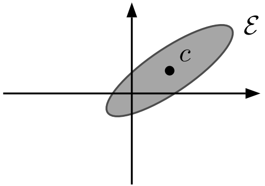

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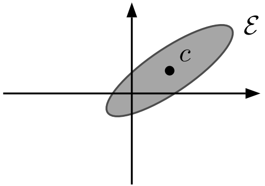

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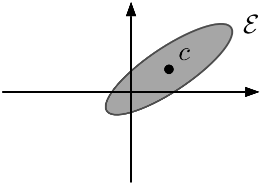

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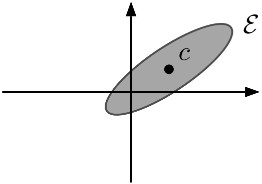

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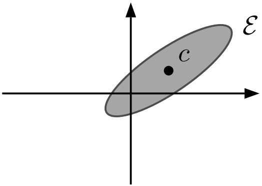

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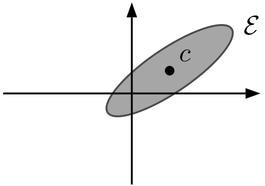

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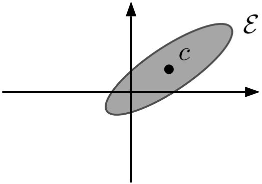

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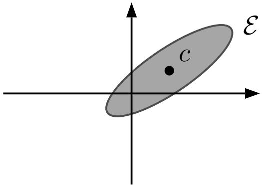

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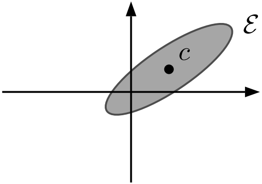

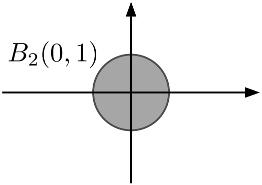

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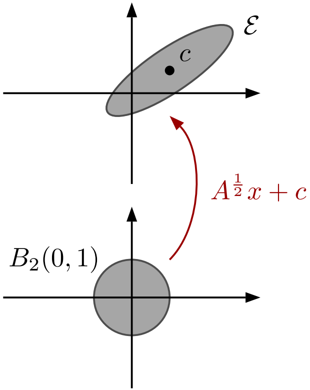

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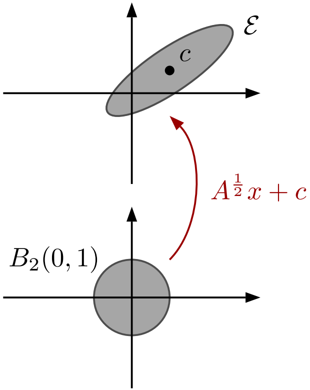

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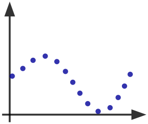

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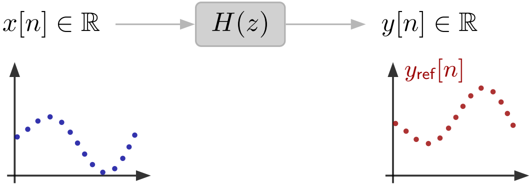

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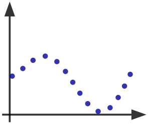

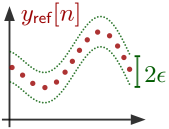

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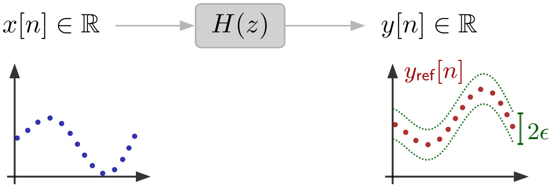

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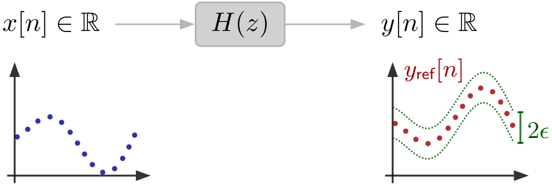

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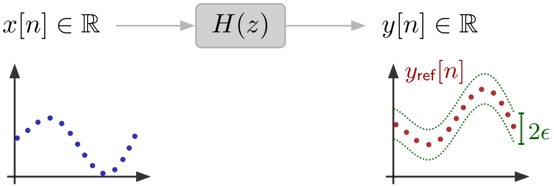

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Matrix form:

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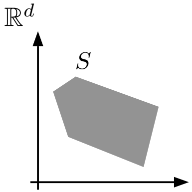

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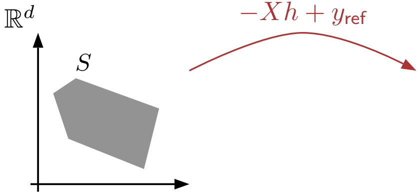

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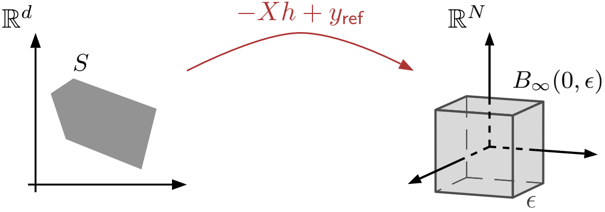

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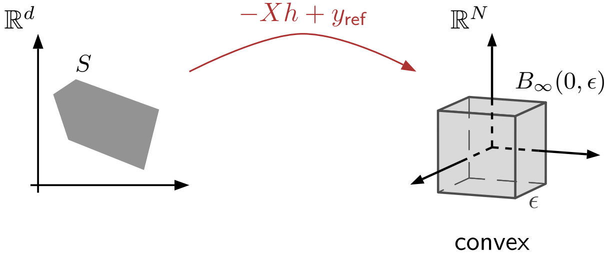

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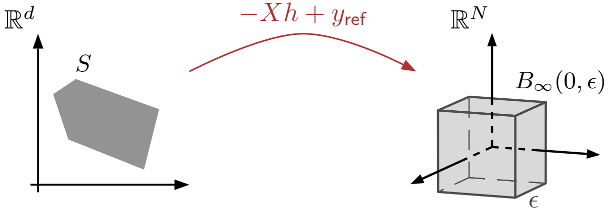

convex $\Longleftarrow$ convex

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

Convex functions

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$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

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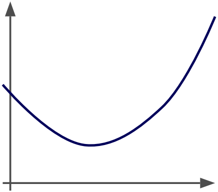

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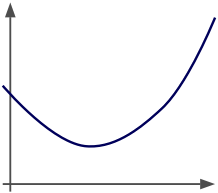

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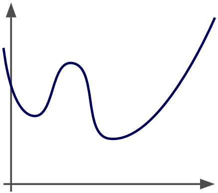

nonconvex

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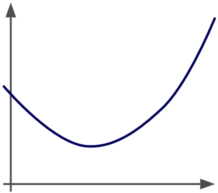

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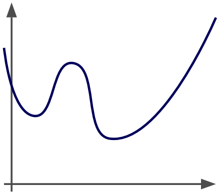

nonconvex

Definition:

f : dom $f \subseteq \mathbb{R}^n \to \mathbb{R}$ is *convex* when for any $x, y \in$ dom f,

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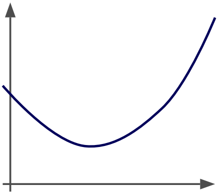

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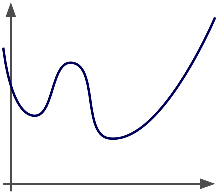

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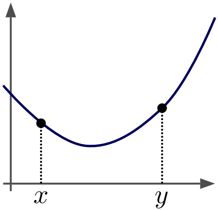

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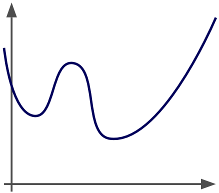

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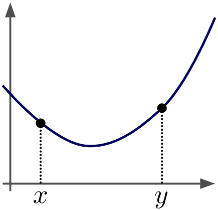

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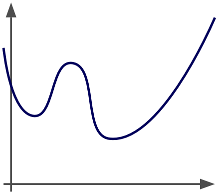

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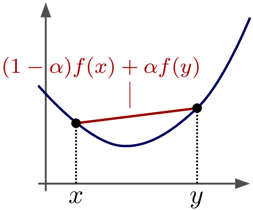

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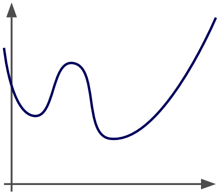

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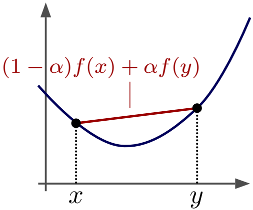

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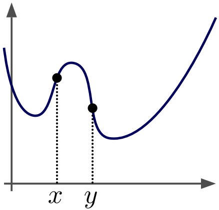

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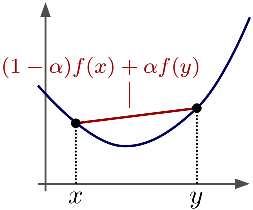

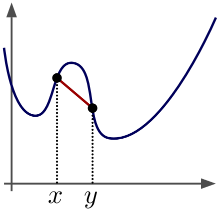

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How to identify convex functions?

vocabulary $+$ **grammar**

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vocabulary	+	**grammar**
definition		*operations preserving convexity*
differentiability conds.		
1D convexity		

Convexity under differentiability

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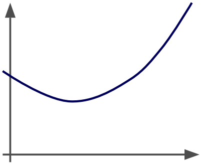

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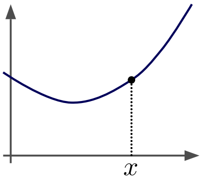

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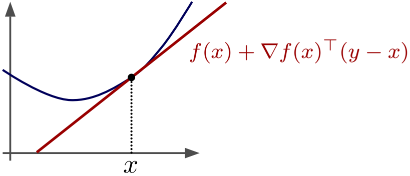

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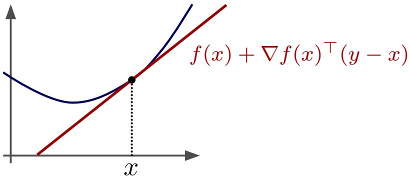

- When f is twice-differentiable,

$$\nabla^2 f(x) \succeq 0\,, \quad \forall_{x\in \mathsf{dom}\, f}$$

Examples

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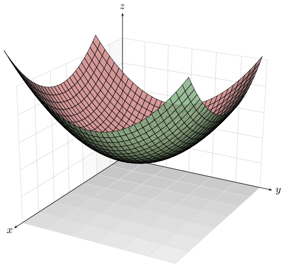

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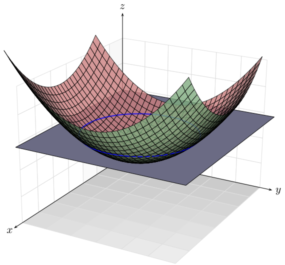

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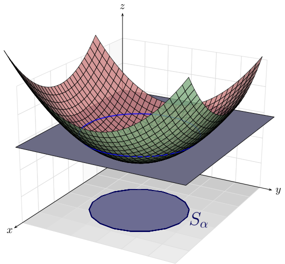

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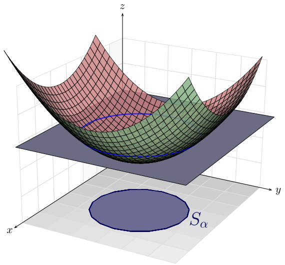

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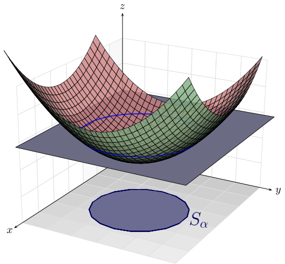

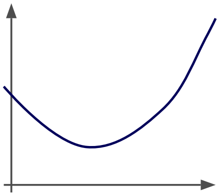

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$$f\big((1-\alpha)x + \alpha y\big) \le (1-\alpha)f(x) + \alpha f(y)\,, \qquad \forall_{x,y \in \mathsf{dom}\, f}\,, \;\; \alpha \in [0,1]$$

Equivalent statements

- $g(t) = f(x+ty) \quad (g : \mathbb{R} \to \mathbb{R})$ is convex for all $x + ty \in \mathsf{dom}\, f$
- Sublevel sets $S_\alpha := \{x \,:\, f(x) \le \alpha\}$ are convex for all $\alpha \in \mathbb{R}$
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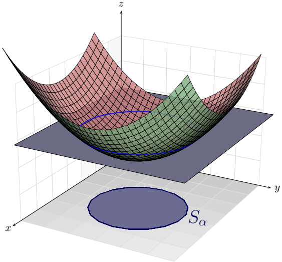

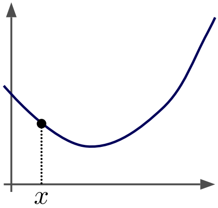

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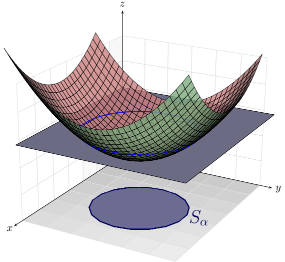

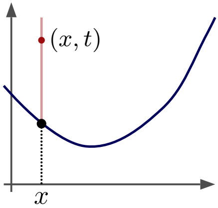

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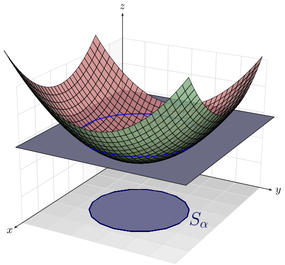

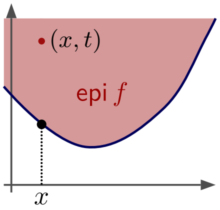

How to identify convex functions?

vocabulary + **grammar**

vocabulary		**grammar**
definition		*operations preserving convexity*
differentiability conds.		
1D convexity		

How to identify convex functions?

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operations preserving convexity

Operations preserving convexity

Operations preserving convexity

Nonnegative weighted sums

Operations preserving convexity

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$$\begin{cases} f_1, \ldots, f_m \ : \text{convex} \\ w_1, \ldots, w_m \geq 0 \end{cases}$$

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Nonnegative weighted sums

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Precomposition with affine maps

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Precomposition with affine maps

$$f \text{ : convex}$$

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Examples

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Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

Convex optimization problems

Convex optimization problems

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0\,, \quad i = 1, \ldots, m \\ & h_i(x) = 0\,, \quad i = 1, \ldots, p \end{aligned}$$

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A minimizer: $$x^\star \in \underset{x}{\text{argmin}} \; f(x) \qquad \text{s.t.} \quad x \in \Omega$$

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Theorem

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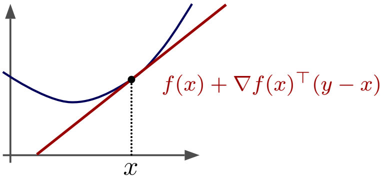

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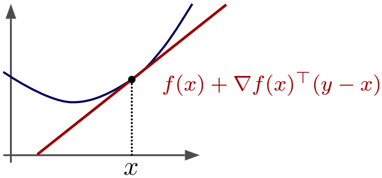

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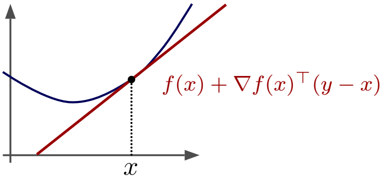

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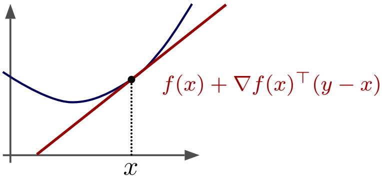

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$\square$

Equivalence between optimization problems

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$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X \end{array} \qquad \text{(P1)}$$

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(P1) and (P2) are *equivalent* when

- Given a solution $x^\star$ of (P1) we can obtain a solution $y^\star$ of (P2)
- Given a solution $y^\star$ of (P2) we can obtain a solution $x^\star$ of (P1)

Examples

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$$\underset{x}{\text{maximize}} \quad f(x)$$

Examples

$$\underset{x}{\text{maximize}} \quad f(x) \qquad \Longleftrightarrow \qquad \underset{x}{\text{minimize}} \quad -f(x)$$

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Examples

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Examples

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$$\underset{x}{\text{minimize}} \quad a\, f(x) + b$$

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$$> 0$$

Examples

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if $\text{im}\, f \subseteq \text{dom}\, g$

and $g \circ f$: convex

Air Traffic Control

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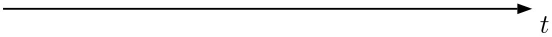

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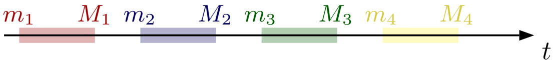

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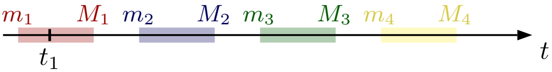

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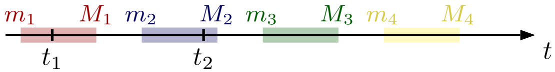

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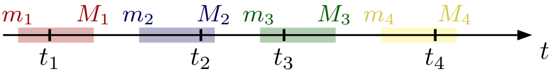

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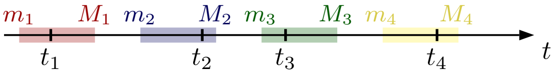

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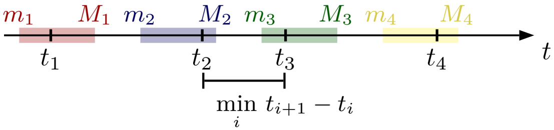

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Convex

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$$A = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & -1 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -m_1 \\ M_1 \\ \vdots \\ -m_n \\ M_n \end{bmatrix}$$

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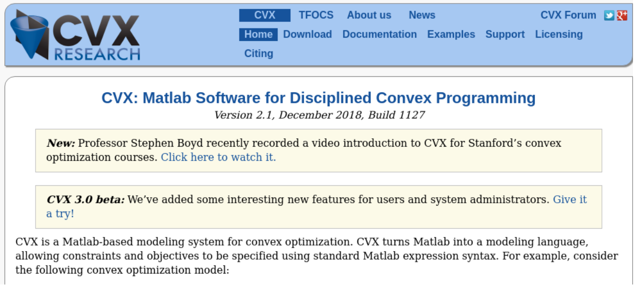
CVX (cvxr.com/cvx) manipulates and solves *convex* problems



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```
cvx_begin
    variables t1 t2 t3 t4 t5;
    maximize( min([t2-t1, t3-t2, t4-t3, t5-t4]) );
    subject to
        1 <= t1 <= 2;
        3 <= t2 <= 4;
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$$(t_1^\star,\ t_2^\star,\ t_3^\star,\ t_4^\star,\ t_5^\star) = (1,\ 3.25,\ 5.5,\ 7.75,\ 10)$$

Portfolio Optimization

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Convex Quadratic Program (QP)

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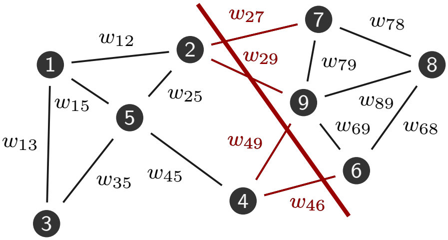
Convex QP

MAXCUT



value of cut: $w_{27} + w_{29} + w_{49} + w_{46}$

Cut: set of edges whose removal splits the graph into two

MAXCUT problem: find the cut with maximum weight

$$
\begin{array}{ll}
\underset{x \in \mathbb{R}^n}{\text{maximize}} & \displaystyle \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{1 - x_i x_j}{2} \\
\text{subject to} & x_i \in \{-1, 1\}, \qquad i = 1, \ldots, n\,.
\end{array}
$$

$$
\begin{aligned}
p^\star = \quad \max_{x} \quad & \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\,\frac{1 - x_i x_j}{2} \\
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Convex Semi-Definite Program (SDP)

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It can be shown that

$$d^\star \,\geq\, p^\star \,\geq\, \mathbb{E}[C] \,\geq\, 0.87856\, d^\star$$

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Large-scale problems & real-time solutions require tailored solvers

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For $n = 5000$ and $m = 500$,

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For $n = 5000$ and $m = 500$,

- CVX: 56.16 s
- SPGL1: 0.82 s (tailored solver)

Optimization Algorithms

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  Block coordinate descent

  Newton (and approximations)

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And now, *neural networks!*

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Example

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \log \sum_{i=1}^{m} \exp\left(a_i^\top x + b\right)$$

randomly generated data with $m = 500$, $n = 50$

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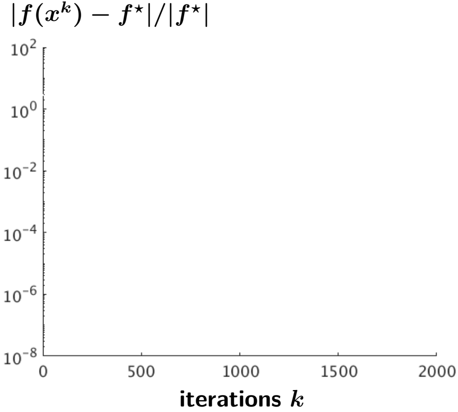

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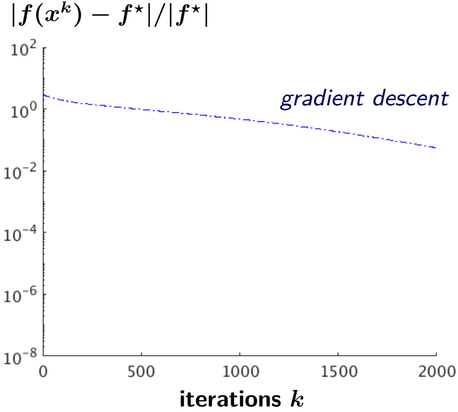

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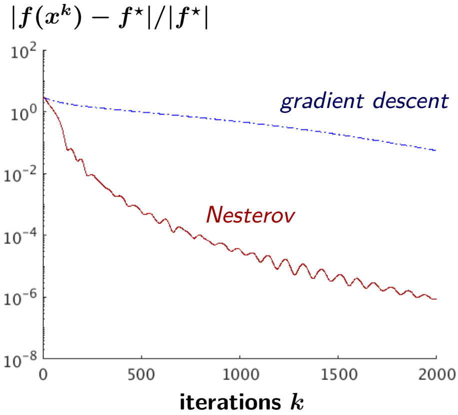

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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The a_i's will denote the rows of $A = \begin{bmatrix} — & a_1^\top & — \\ & \vdots & \\ — & a_m^\top & — \end{bmatrix} \in \mathbb{R}^{m \times n}$

Maximum likelihood

$$\widehat{x}_{\mathsf{ML}} \;\in\; \arg\min_x \; -\log \prod_{i=1}^{m} f_V(y_i - a_i^\top x)$$

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Convex

Maximum likelihood

$$\widehat{x}_{\mathsf{ML}} \in \arg\min_x -\sum_{i=1}^{m} \log f_V(y_i - a_i^\top x)$$

Laplacian:

Maximum likelihood

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Convex

Maximum likelihood

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feasibility problem (convex)

Example with a discrete RV

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$Y \in \mathbb{N}$: $\#$ of traffic accidents in a given period

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e.g., $U_1 =$ traffic flow during the period, $U_2 =$ rainfall, ...

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Goal:

Given m independent observations $\left\{\left(U^{(i)}, Y^{(i)}\right)\right\}_{i=1}^{m}$, estimate a and b.

Example with a discrete RV

Joint probability mass function: $p_{YU}(y, u\,;\, a, b) = \frac{e^{-(a^\top u+b)}\,(a^\top u+b)^y}{y\,!}$

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Convex

Maximum a posteriori (MAP)

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$Y \in \mathbb{R}^m$: random vector w/ density $f_Y(y)$

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Maximum a posteriori (MAP)

Example: linear measurement model

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- Observations: $Y_i = a_i^\top X + V_i, \;\; i = 1, \ldots, m$

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  $f_{Y_1 \cdots Y_m|X}(y_1, \ldots, y_m|x)$

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- Random vector to estimate in $\mathbb{R}^n$: $X \sim \mathcal{N}\big(\overline{x}, \Sigma\big)$

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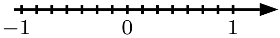

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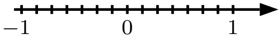

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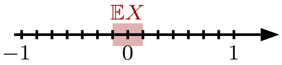

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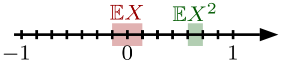

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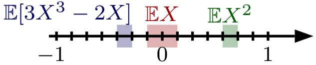

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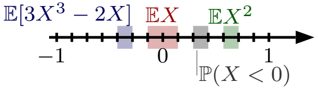

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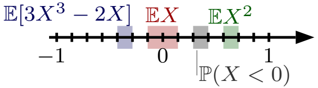

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Find a distribution satisfying these constraints & with maximum entropy

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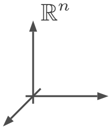

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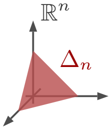

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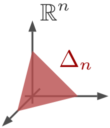

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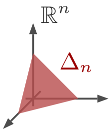

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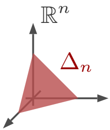

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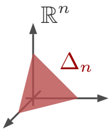

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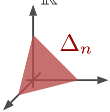

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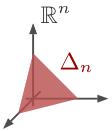

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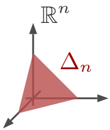

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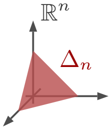

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All constraints are *linear inequalities* in p!

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Convex because $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} x \log x = \frac{1}{x} > 0$, for all $x > 0$

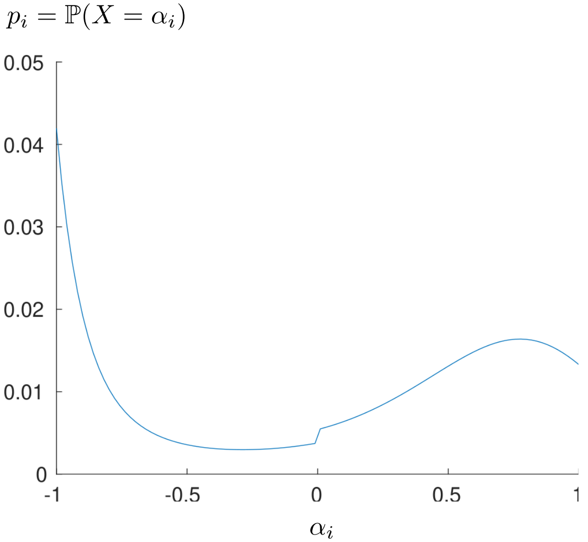
Optimization problem: *(convex)*

$$\begin{array}{ll} \underset{p \in \mathbb{R}^{100}}{\text{minimize}} & \sum_{i=1}^{n} p_i \log p_i \\ \text{subject to} & -0.1 \leq \alpha^\top p \leq 0.1 \\ & 0.5 \leq \beta^\top p \leq 0.6 \\ & -0.3 \leq \gamma^\top p \leq -0.2 \\ & 0.3 \leq \sigma^\top p \leq 0.4 \end{array}$$

Nonparametric estimation

```
n = 100;
alpha = linspace(-1,1,n)';
cvx_begin
    variable p(n,1);
    minimize( -sum( entr( p ) ) );
    subject to
        p >= 0;
        ones(1, n)*p == 1;
        -0.1 <= alpha'*p <= 0.1;
        0.5 <= (alpha.^2)'*p <= 0.6;
        -0.3 <= (3*alpha.^3 - 2*alpha)'*p <= -0.2;
        0.3 <= (alpha < 0)'*p <= 0.4;
cvx_end
```

Nonparametric estimation



Hypothesis testing & optimal detection

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X : discrete random variable w/ probability mass function (pmf) p_θ

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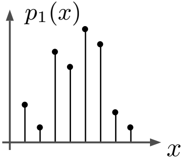

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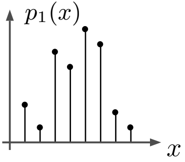

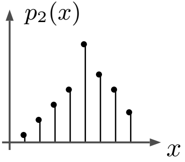

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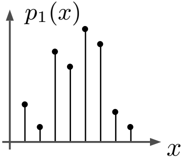

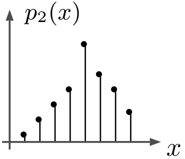

$$P = \begin{bmatrix} \mathbb{P}(X=1\,|\,\theta=1) & \mathbb{P}(X=1\,|\,\theta=2) & \cdots & \mathbb{P}(X=1\,|\,\theta=m) \\ \mathbb{P}(X=2\,|\,\theta=1) & \mathbb{P}(X=2\,|\,\theta=2) & \cdots & \mathbb{P}(X=2\,|\,\theta=m) \\ \vdots & \vdots & & \vdots \\ \mathbb{P}(X=n\,|\,\theta=1) & \mathbb{P}(X=n\,|\,\theta=2) & \cdots & \mathbb{P}(X=n\,|\,\theta=m) \end{bmatrix} \in \mathbb{R}^{n\times m}$$

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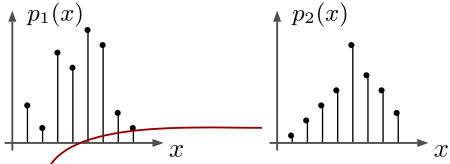

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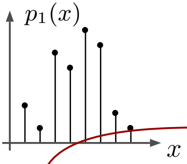

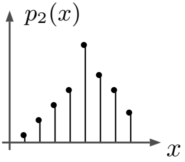

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Goal: Estimate θ based on an observation of X

Detector:

Detector: $\Psi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$

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Example: maximum likelihood detector

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Each column $t_i \in \mathbb{R}^m$ satisfies $1_m^\top t_i = 1$, $t_i \geq 0$

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Each column $t_i \in \mathbb{R}^m$ satisfies $1_m^\top t_i = 1$, $t_i \geq 0$

If each t_i is a canonical vector $(0,\ldots,1,\ldots,0)$, then T is deterministic

Detection probability matrix:

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\end{aligned}
$$

Goal: Design T such that the D_{ij}'s, for $i \neq j$, are as small as possible

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Multi-objective optimization

Minimax detector

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Note that

$$\mathbb{P}\big(\widehat{\theta} \neq j \,|\, \theta = j\big)$$

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$$\begin{array}{ll} \underset{T \in \mathbb{R}^{m \times n}}{\text{minimize}} & \displaystyle\max_{j=i,\ldots,m} \; 1 - D_{jj}(T) \\ \text{subject to} & T^\top 1_m = 1_n \,, \;\; T \geq 0 \end{array}$$

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Convex

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  where c_i: ith column of $C := WP^\top$

Pareto-optimal detectors

$$
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Decouples into n independent optimization problems: for column i,

Pareto-optimal detectors

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\end{array}
$$

$$
\iff \qquad
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$$

Decouples into n independent optimization problems: for column i,

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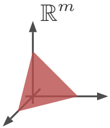

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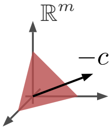

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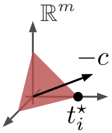

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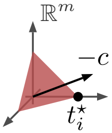

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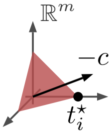

$$t_i^\star = (0,\ldots,1,\ldots,0) \qquad \Longrightarrow \qquad T \text{ is deterministic}$$

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X is generated by either p_1 $(\theta = 1)$ or p_2 $(\theta = 2)$

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Therefore,

$$c_i^\top t_i = t_{1i}\,W_{12}\,\mathbb{P}(X=i\,|\,\theta=2) + t_{2i}\,W_{21}\,\mathbb{P}(X=i\,|\,\theta=1)$$

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Neyman-Pearson lemma:

For each $\alpha > 0$, the likelihood-ratio test yields a (deterministic) Pareto-optimal detector.

Deterministic vs randomized detectors

$$P = \begin{bmatrix} 0.70 & 0.10 \\ 0.20 & 0.10 \\ 0.05 & 0.70 \\ 0.05 & 0.10 \end{bmatrix}$$

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Minimax detector (random):

$$T_{\mathsf{MM}} = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1/3 & 1 & 1 \end{bmatrix}$$

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Receiver operating characteristic (ROC)

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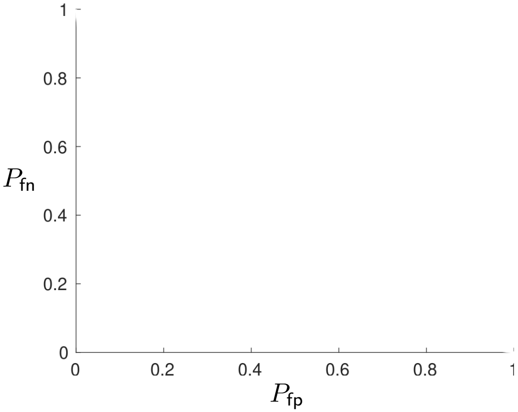

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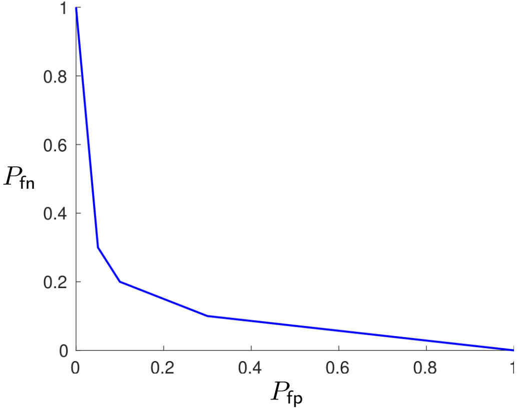

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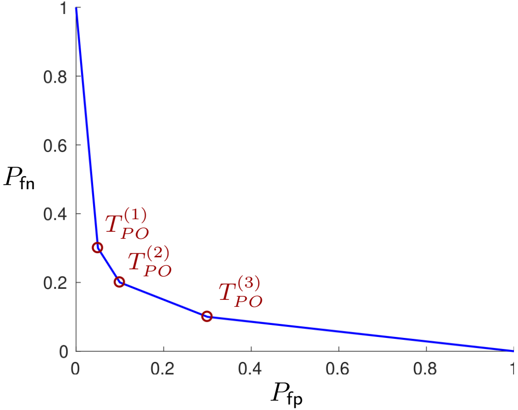

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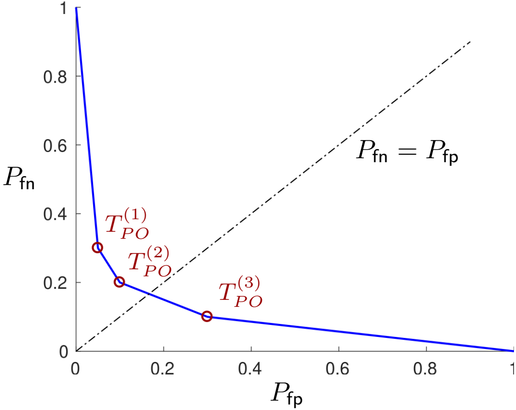

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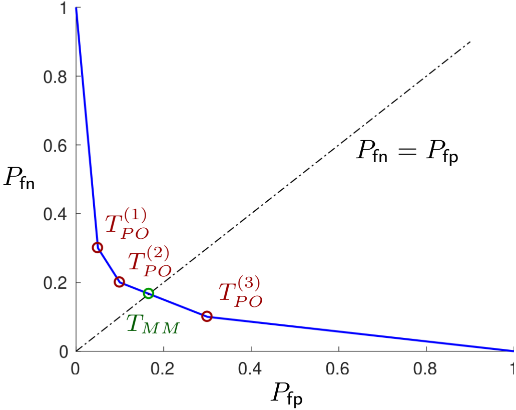

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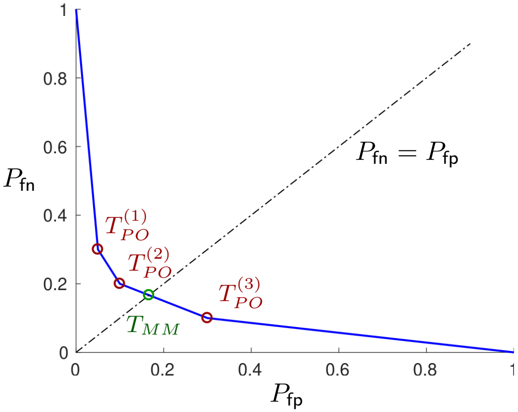

Minimax estimator has $(P_{\mathsf{fp}}, P_{\mathsf{fn}}) = (\frac{1}{6}, \frac{1}{6})$ and outperforms any deterministic estimator

Conclusions

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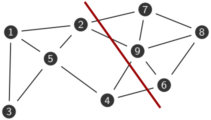

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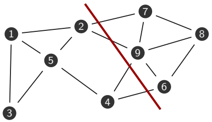

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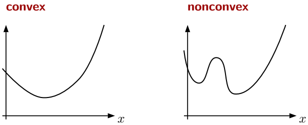

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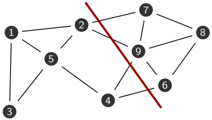

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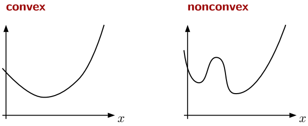
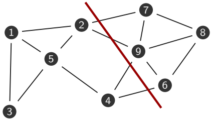


Conclusions



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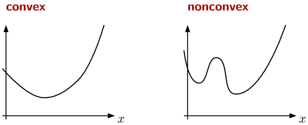
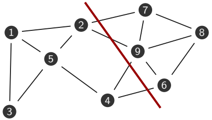
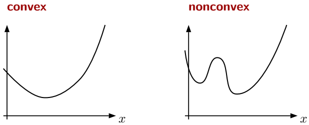
- *Didn't cover* optimality conditions & theory



Conclusions



- Optimization problems arise in many areas
- Essential to distinguish easy *(convex)* from hard *(nonconvex)* probs
- Use CVX/CVXPY/Convex for small-scale problems
- *Didn't cover* optimality conditions & theory
- *Didn't cover* optimization algorithms



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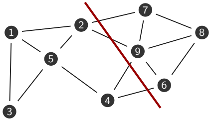

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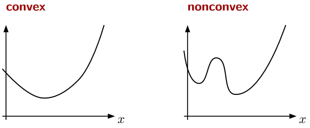
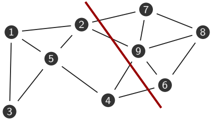


- Statistical estimation:

  Find parameters of distributions (ML/MAP)

  Entire distributions (nonparametric)

Conclusions



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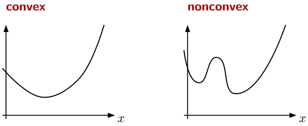

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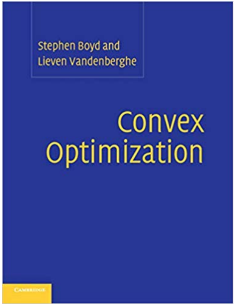
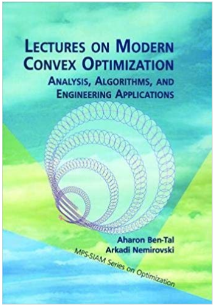
  Find parameters of distributions (ML/MAP)

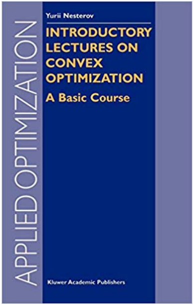
  Entire distributions (nonparametric)

- Multiple hypothesis testing via optimization

References and Resources







Lectures:

- web.stanford.edu/∼boyd/cvxbook/
- users.isr.ist.utl.pt/∼jxavier/NonlinearOptimization18799-2018
- www.seas.ucla.edu/∼vandenbe/ee236c