Widely Linear State Space Models for Frequency Estimation in Unbalanced Three-Phase Systems

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***Abstract*—A novel technique for the online frequency estimation of three-phase power systems using the widely linear (augmented) complex least mean square (ACLMS) algorithm has recently been proposed, and was shown to achieve significantly better estimates than conventional complex least mean square (CLMS) algorithm based frequency estimation under unbalanced system conditions. In this paper, we consider the frequency estimation problem from the state space point of view, and show that the augmented complex Kalman filter (ACKF) offers significantly better performance than ACLMS.**

***Index Terms*—Complex Kalman filter; widely linear model; complex circularity; frequency estimation; smart grid**

I. Introduction

Accurate frequency estimation is essential for the protection of power systems as well as for providing desired power quality. Variation of the system frequency from its normal value can indicate the occurrence of unexpected system conditions which might require corrective action to be taken, and as such, frequency tracking and estimation has received much attention. A number of frequency estimation algorithms have been proposed; some of the most established ones include least square adaptive filters [1], state space estimation using Kalman filters [2] and Fourier transform based approaches. However, these techniques are either only suitable for balanced systems, that is, systems with line voltages of equal amplitudes, or have been designed for single-phase systems, and hence cannot fully characterise three-phase power systems, where the line-to-line voltages need to be taken into account.

The linear mapping, known as Clarke's $\alpha\beta$ transformation is instrumental in providing a unified framework for dealing with all three-phase voltages simultaneously, and has lead to the development of a number of complex valued frequency estimation techniques. However, these techniques have proven suboptimal for unbalanced voltage conditions, for example, when the three line voltages have different amplitudes, resulting in an oscillatory estimation error at twice the system frequency. This problem can be attributed to the use of the standard, strictly linear, complex estimation model that does not fully capture the second order statistics of complex signals.

Recent advances in the so called 'augmented complex statistics' [3] have highlighted that for a general (improper) complex vector $\mathbf{x}$, second order statistics based on the covariance $\mathbf{R_x} = E\{\mathbf{x}\mathbf{x}^H\}$ is inadequate and that the pseudocovariance $\mathbf{P_x} = E\{\mathbf{x}\mathbf{x}^T\}$ is also required to fully capture the second order statistics. To introduce an optimal second order estimator for the generality of complex signals, consider first the mean square estimator (MSE) of a real valued random vector $\mathbf{y}$ in terms of an observed real vector $\mathbf{x}$, that is, $\hat{\mathbf{y}} = E\{\mathbf{y}|\mathbf{x}\}$. For zero-mean, jointly normal $\mathbf{y}$ and $\mathbf{x}$, the optimal estimator is linear, that is

$$\hat{\mathbf{y}} = \mathbf{H}\mathbf{x} \tag{1}$$

where $\mathbf{H}$ is a coefficient matrix. Standard, 'strictly linear' estimation in $\mathbb{C}$ assumes the same model but with complex valued $\mathbf{y}, \mathbf{x}$, and $\mathbf{H}$. However, observe that both the real $\mathbf{y}_r$ and imaginary $\mathbf{y}_i$ parts of the vector $\mathbf{y}$ are real valued, and

$$\hat{\mathbf{y}}_r = E\{\mathbf{y}_r|\mathbf{x}_r, \mathbf{x}_i\} \qquad \hat{\mathbf{y}}_i = E\{\mathbf{y}_i|\mathbf{x}_r, \mathbf{x}_i\} \tag{2}$$

Substituting $\mathbf{x}_r = (\mathbf{x} + \mathbf{x}^*)/2$ and $\mathbf{x}_i = (\mathbf{x} - \mathbf{x}^*)/2\jmath$ yields

$$\hat{\mathbf{y}}_r = E\{\mathbf{y}_r|\mathbf{x}, \mathbf{x}^*\} \qquad \hat{\mathbf{y}}_i = E\{\mathbf{y}_i|\mathbf{x}, \mathbf{x}^*\} \tag{3}$$

and using (1) we obtain the *widely linear* complex estimator

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{G}\mathbf{x}^* = \mathbf{W}\mathbf{x}^a \tag{4}$$

where the matrix $\mathbf{W}$ comprises the coefficient matrices $\mathbf{H}$ and $\mathbf{G}$, and $\mathbf{x}^a = [\mathbf{x}^T, \mathbf{x}^H]^T$ is the 'augmented' input vector. The full second order information is thus contained in the augmented covariance matrix

$$\mathbf{R}^a_\mathbf{x} = E\{\mathbf{x}^a\mathbf{x}^{aH}\} = \begin{bmatrix} \mathbf{R_x} & \mathbf{P_x} \\ \mathbf{P}^*_\mathbf{x} & \mathbf{R}^*_\mathbf{x} \end{bmatrix} \tag{5}$$

It has recently been shown in [4] that under unbalanced power system conditions, the system becomes noncircular, and the widely linear model is required for accurate system representation; for this purpose, the work in [4] [5] developed widely linear (augmented) complex least mean square (ACLMS) algorithm for frequency estimation [6].

We here introduce the widely linear complex Kalman filter (ACKF) [7] with the aim of fully utilising the complete second order statistics of complex signals [6] [8]. To that end, we revisit the widely linear frequency estimation approach from a state space point of view, and show that ACKF offers more accurate estimates and faster convergence than stochastic gradient based algorithms. Illustrative simulations on unbalanced synthetic and real world data support the analysis.

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II. The Widely Linear (augmented) complex Kalman filter (ACKF)

The Kalman filter is a sequential state estimator for linear dynamical systems. In its basic form, it employs the linear state space model given by

$$\begin{aligned} \mathbf{x}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{u}_{k-1} \quad (6) \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{n}_k \quad (7) \end{aligned}$$

where $\mathbf{x}_k$ is the state to be estimated at time instant k, $\mathbf{y}_k$ is the noisy observation, the vectors $\mathbf{u}_k$ and $\mathbf{n}_k$ are the zero-mean state and measurement noises, with covariance matrices $\mathbf{R}_{\mathbf{u},k}$ and $\mathbf{R}_{\mathbf{n},k}$ and pseudocovariances $\mathbf{P}_{\mathbf{u},k}$ and $\mathbf{P}_{\mathbf{n},k}$, while $\mathbf{F}_k$ and $\mathbf{H}_k$ are the state transition matrix and observation matrix. Based on (4), the corresponding widely linear state space model is defined as

$$\begin{aligned} \mathbf{x}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{A}_{k-1}\mathbf{x}^*_{k-1} + \mathbf{u}_{k-1} \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{x}^*_k + \mathbf{n}_k \end{aligned}$$

and can be expressed in more compact form using so called "augmented" vectors, such that [6]

$$\begin{aligned} \mathbf{x}^a_k &= \mathbf{F}^a_{k-1}\mathbf{x}^a_{k-1} + \mathbf{u}^a_{k-1} \quad (8) \\ \mathbf{y}^a_k &= \mathbf{H}^a_k\mathbf{x}^a_k + \mathbf{n}^a_k \quad (9) \end{aligned}$$

where $\mathbf{x}^a_k = [\mathbf{x}^T_k, \mathbf{x}^H_k]^T$, $\mathbf{y}^a_k = [\mathbf{y}^T_k, \mathbf{y}^H_k]^T$, $\mathbf{F}^a_k = \begin{bmatrix} \mathbf{F}_k & \mathbf{A}_k \\ \mathbf{A}^*_k & \mathbf{F}^*_k \end{bmatrix}$ and $\mathbf{H}^a_k = \begin{bmatrix} \mathbf{H}_k & \mathbf{B}_k \\ \mathbf{B}^*_k & \mathbf{H}^*_k \end{bmatrix}$.

The coefficient matrices, $\mathbf{A}$ and $\mathbf{B}$, determine whether the state and observation equations are strictly linear or widely linear. For $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$, the state space is strictly linear, however, the augmented state space representation should still be preferred over the linear state space, if the state and observation noises are second order noncircular. Consider next the augmented covariance matrices of $\mathbf{u}^a_k = [\mathbf{u}^T_k, \mathbf{u}^H_k]^T$ and $\mathbf{n}^a_k = [\mathbf{n}^T_k, \mathbf{n}^H_k]^T$, that is

$$\begin{aligned} \mathbf{R}^a_{\mathbf{u},k} &= E\{\mathbf{u}^a_k\mathbf{u}^{aH}_k\} = \begin{bmatrix} \mathbf{R}_{\mathbf{u},k} & \mathbf{P}_{\mathbf{u},k} \\ \mathbf{P}^*_{\mathbf{u},k} & \mathbf{R}^*_{\mathbf{u},k} \end{bmatrix} \quad (10) \\ \mathbf{R}^a_{\mathbf{n},k} &= E\{\mathbf{n}^a_k\mathbf{n}^{aH}_k\} = \begin{bmatrix} \mathbf{R}_{\mathbf{n},k} & \mathbf{P}_{\mathbf{n},k} \\ \mathbf{P}^*_{\mathbf{n},k} & \mathbf{R}^*_{\mathbf{n},k} \end{bmatrix} \quad (11) \end{aligned}$$

where the pseudocovariances are naturally catered for. The widely linear estimate $\widehat{\mathbf{x}}^a_{k|k}$ of $\mathbf{x}^a_k$ based on the observations $\{\mathbf{y}^a_1, \mathbf{y}^a_2, ..., \mathbf{y}^a_k\}$ can be computed sequentially using the ACKF, which is summarised in Algorithm 1.

The mean square error (MSE) difference between conventional complex Kalman filter (CCKF) and widely linear (augmented) complex Kalman filter (ACKF), after some tedious algebraic manipulations, can be written as [3]

$$\begin{aligned} \Delta\mathbf{M}_k = &(\mathbf{P}_{\mathbf{xz},k,k} - \mathbf{R}_{\mathbf{xz},k,k}\mathbf{R}^{-1}_{\mathbf{z},k}\mathbf{P}_{\mathbf{z},k}) \\ &\times(\mathbf{R}^*_{\mathbf{z},k} - \mathbf{P}^*_{\mathbf{z},k}\mathbf{R}^{-1}_{\mathbf{z},k}\mathbf{P}_{\mathbf{z},k})^{-1} \\ &\times(\mathbf{P}_{\mathbf{xz},k,k} - \mathbf{R}_{\mathbf{xz},k,k}\mathbf{R}^{-1}_{\mathbf{z},k}\mathbf{P}_{\mathbf{z},k})^H \quad (17) \end{aligned}$$

Algorithm 1. The augmented complex Kalman filter (ACKF)

Initialise with:

$$\begin{aligned} \widehat{\mathbf{x}}^a_{0|0} &= E\{\mathbf{x}^a_0\} \\ \mathbf{M}^a_{0|0} &= E\{(\mathbf{x}^a_0 - E\{\mathbf{x}^a_0\})(\mathbf{x}^a_0 - E\{\mathbf{x}^a_0\})^H\} \end{aligned}$$

State Prediction:

$$\widehat{\mathbf{x}}^a_{k|k-1} = \mathbf{F}^a_{k-1}\widehat{\mathbf{x}}^a_{k-1|k-1} \quad (12)$$

Prediction Covariance Matrix:

$$\mathbf{M}^a_{k|k-1} = \mathbf{F}^a_{k-1}\mathbf{M}^a_{k-1|k-1}\mathbf{F}^{aH}_{k-1} + \mathbf{R}^a_{\mathbf{u},k-1} \quad (13)$$

Kalman Gain:

$$\mathbf{G}^a_k = \mathbf{M}^a_{k|k-1}\mathbf{H}^{aH}_k[\mathbf{H}^a_k\mathbf{M}^a_{k|k-1}\mathbf{H}^{aH}_k + \mathbf{R}^a_{\mathbf{n},k}]^{-1} \quad (14)$$

State Update:

$$\widehat{\mathbf{x}}^a_{k|k} = \widehat{\mathbf{x}}^a_{k|k-1} + \mathbf{G}^a_k(\mathbf{y}^a_k - \mathbf{H}^a_k\widehat{\mathbf{x}}^a_{k|k-1}) \quad (15)$$

Covariance Matrix:

$$\mathbf{M}^a_{k|k} = (\mathbf{I} - \mathbf{G}^a_k\mathbf{H}^a_k)\mathbf{M}^a_{k|k-1} \quad (16)$$

where $\mathbf{z}_k = [\mathbf{y}^T_1, \mathbf{y}^T_2, ..., \mathbf{y}^T_k]^T$ is the observation sequence with covariance and pseudocovariance $\mathbf{R}_{\mathbf{z},k}$ and $\mathbf{P}_{\mathbf{z},k}$ respectively, while, $\mathbf{R}_{\mathbf{xz},k,k} = E\{(\mathbf{x}_k - E\{\mathbf{x}_k\})(\mathbf{z}_k - E\{\mathbf{z}_k\})^H\}$ and $\mathbf{P}_{\mathbf{xz},k,k} = E\{(\mathbf{x}_k - E\{\mathbf{x}_k\})(\mathbf{z}_k - E\{\mathbf{z}_k\})^T\}$ are the cross-correlation and pseudocorrelation between the state and observation sequence.

Remark 1: The expression (17) is always positive semidefinite since the matrix $(\mathbf{R}^*_{\mathbf{z},k} - \mathbf{P}^*_{\mathbf{z},k}\mathbf{R}^{-1}_{\mathbf{z},k}\mathbf{P}_{\mathbf{z},k})$ is positive definite, and consequently $\Delta\mathbf{M}_k = \mathbf{0}$ only when $(\mathbf{P}_{\mathbf{xz},k,k} - \mathbf{R}^{-1}_{\mathbf{xz},k,k}\mathbf{R}_{\mathbf{z},k}\mathbf{P}_{\mathbf{z},k}) = \mathbf{0}$. Therefore, the ACKF always has the same or better MSE performance than CCKF.

Remark 2: The CCKF and ACKF are equivalent, that is $\Delta\mathbf{M}_k = \mathbf{0}$, if and only if the state and observation noises are both circular and the state and observation equations are both strictly linear.

III. Widely Linear Frequency Estimation

The noise free three-phase voltages can be defined as

$$\begin{aligned} v_{a,k} &= V_{a,k}\cos(\omega kT + \phi) \\ v_{b,k} &= V_{b,k}\cos(\omega kT + \phi - 2\pi/3) \\ v_{c,k} &= V_{c,k}\cos(\omega kT + \phi + 2\pi/3) \quad (18) \end{aligned}$$

where $V_{a,k}$, $V_{b,k}$ and $V_{c,k}$ are the amplitudes of the of the three-phase voltages at time instant k, $\omega = 2\pi f$ is the angular frequency with f being the system frequency, T is the sampling interval and ϕ is the phase of the fundamental component. Clarke's transformation, given by

$$\begin{bmatrix} v_{0,k} \\ v_{\alpha,k} \\ v_{\beta,k} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_{a,k} \\ v_{b,k} \\ v_{c,k} \end{bmatrix}, \quad (19)$$

is normally used to map the three-phase voltages onto a new domain where they can be conveniently represented by a scalar complex signal. In (19), the zero-sequence $v_{0,k}$ vanishes for balanced systems, that is, when $V_{a,k} = V_{b,k} = V_{c,k}$, while $v_{\alpha,k} = A_k \cos(\omega kT + \phi)$ and $v_{\beta,k} = A_k \cos(\omega kT + \phi + \frac{\pi}{2})$ are orthogonal signals. In practice, the zero-sequence $v_{0,k}$ is ignored, and only v_α and v_β are used to form the complex output which models the system, that is

$$v_k = v_{\alpha,k} + jv_{\beta,k} = A_k e^{j(\omega kT+\phi)} = v_{k-1}e^{j\omega T} \quad (20)$$

where the usual assumption $A_k \approx A_{k-1}$ is utilised. From a state space point of view, the system can be modeled as in Algorithm 2. The state x_k is used to estimate $e^{j\omega T}$ which contains the instantaneous frequency f and v_k is the observation, while u_k and n_k are the state and observations noises. The three-phase system frequency is derived from the state x as

Algorithm 2. State Space 1 - Linear (SS1-L)

$$\text{state equation:} \quad x_k = x_{k-1} + u_{k-1} \quad (21)$$

$$\text{observation equation:} \quad v_k = v_{k-1}x_k + n_k \quad (22)$$

$$\hat{f}_k = \frac{1}{2\pi T}\arcsin\left(\Im(x_k)\right) \quad (23)$$

where $\Im(\cdot)$ is the imaginary part of a complex quantity.

Based on (20), it is clear that v_k follows a circular trajectory, since the amplitude is invariant with time, and has an angular frequency proportional to the system frequency. However, this circular model does not hold when the power system is operating under abnormal conditions, such as when a voltage sag occurs, in which case the voltage amplitudes $V_{a,k}$, $V_{b,k}$ and $V_{c,k}$ are no longer equal, and the system trajectory becomes noncircular. It was shown in [4] that, in this case, the true system model becomes widely linear, that is

$$\begin{aligned} v_k &= v_{\alpha,k} + jv_{\beta,k} \\ &= A_k e^{j(\omega kT+\phi)} + B_k e^{-j(\omega kT+\phi)} \end{aligned} \quad (24)$$

with

$$\begin{aligned} A_k &= \frac{\sqrt{6}(V_{a,k}+V_{b,k}+V_{c,k})}{6} \\ B_k &= \frac{\sqrt{6}(2V_{a,k}-V_{b,k}-V_{c,k})}{12} - \frac{\sqrt{2}(V_{b,k}-V_{c,k})}{4}j \end{aligned} \quad (25)$$

When the system is balanced and operating under nominal conditions, that is $V_{a,k} = V_{b,k} = V_{c,k}$, the coefficient B_k vanishes and system is accurately characterised by (20), otherwise, the expression in (20) is an inaccurate representation of the system, since the system is noncircular for $B_k \neq 0$ (see Figure 1). Observe that, the expression in (24) characterises the system under both balanced and unbalanced conditions, and can be written recursively as

$$v_k = v_{k-1}h_{k-1} + v^*_{k-1}g_{k-1} \quad (26)$$

The corresponding widely linear (augmented) state space model can be defined as in Algorithm 3, where the state is the vector consisting of the strictly linear weight h_k and conjugate weight g_k, and the observation v_k is widely linear in terms of the previous observation, while, $\mathbf{u}_{h,k}$ and $\mathbf{u}_{g,k}$ are the state noises corresponding to h_k and g_k. The system frequency is computed as

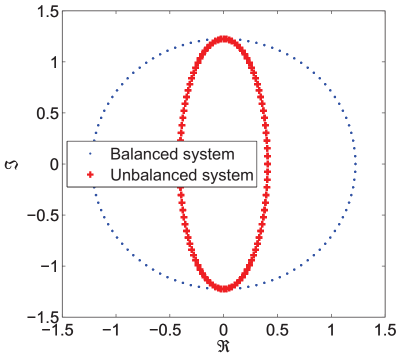

Fig. 1. Geometric view of circularity via a real-imaginary scatter plot of the output voltage v_k. For a balanced system, characterised by $V_{a,k} = V_{b,k} = V_{c,k}$, the trajectory of v_k is circular (dotted circle), while, for unbalanced systems, such as the 100% single-phase voltage sag in phase $v_{a,k}$ illustrated by the ellipse in the figure (+), the trajectory becomes noncircular.

$$\hat{f}_k = \frac{1}{2\pi T}\arcsin\left(\Im(h_k + a_k g_k)\right) \quad (29)$$

with

$$a_k = \frac{-j\Im(h_k) + j\sqrt{\Im^2(h_k) - |g_k|^2}}{g_k}$$

IV. Simulations

The proposed sequential state space algorithms and the stochastic gradient based CLMS and ACLMS were assessed using a 5kHz sampling rate, and were all initialised to 50.5Hz. The linear state space models SS1 was implemented using CCKF, while, model SS2 was implemented using the ACKF.

Figure 2 shows the performances of the considered algorithms for an initially balanced system which became unbalanced after undergoing a Type C voltage sag starting at 0.1s, characterised by a 20% voltage drop and 10^o phase offset on both v_b and v_c, followed by a Type D sag starting at 0.25s, characterised by a 20% voltage drop at line v_a and

Algorithm 3. State Space 2 - Widely Linear (SS2-WL)

state equation:

$$\begin{bmatrix} h_k \\ g_k \\ h^*_k \\ g^*_k \end{bmatrix} = \begin{bmatrix} h_{k-1} \\ g_{k-1} \\ h^*_{k-1} \\ g^*_{k-1} \end{bmatrix} + \begin{bmatrix} u_{h,k-1} \\ u_{g,k-1} \\ u^*_{h,k-1} \\ u^*_{g,k-1} \end{bmatrix} \quad (27)$$

observation equation:

$$v_k = \begin{bmatrix} v_{k-1} & v^*_{k-1} & 0 & 0 \\ 0 & 0 & v^*_{k-1} & v_{k-1} \end{bmatrix} \begin{bmatrix} h_k \\ g_k \\ h^*_k \\ g^*_k \end{bmatrix} + \begin{bmatrix} n_k \\ n^*_k \end{bmatrix} \quad (28)$$

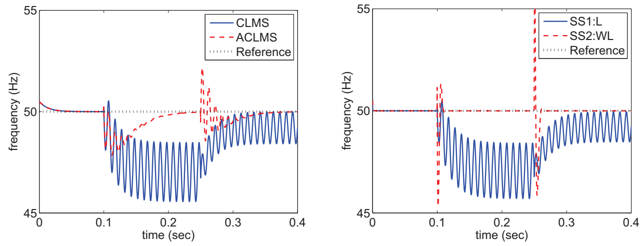

Fig. 2. Frequency estimation for a system which is balanced up to 0.1s, after which the system becomes unbalanced due to the occurrence of voltage sags of differing natures.

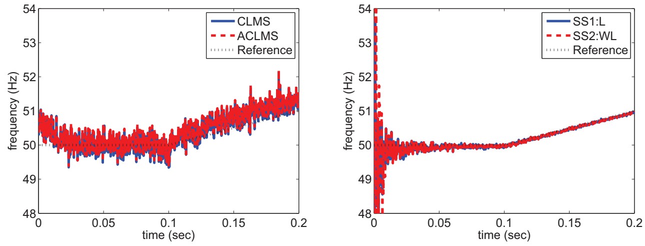

Fig. 3. Frequency estimation for a balanced system that experiences a 10Hz/s frequency rise starting at time 0.1s, all in the presence of white Gaussian noises at 25dB SNR.

a 10% voltage drop on both v_b and v_c with a 5^o phase angle offset. Observe that for unbalanced systems, the widely linear algorithms, ACLMS and SS2, were able to accurately estimate the system frequency, conforming with the analysis, while the strictly linear algorithms, CLMS and SS1, yielded oscillating frequency estimates due to undermodeling of the system. However, both sets of algorithms had similar performances under balanced system conditions (time interval $0 - 0.1$s).

Figure 3 illustrates frequency estimation in the presence of observation noise for a system with rising frequency, typical case of when generation is larger than consumption. The CLMS and ACLMS estimates were inaccurate, while the Kalman filters were able to accurately track system frequency.

The last set of simulations considers frequency estimation for a real-world power system with nominal frequency of 50Hz (sampled at a rate of 1kHz). Figure 4 shows the results for an unbalanced system (a single-phase short with earth), where the theoretical and practical superiority of the algorithms based on the widely linear models, ACLMS and SS2, compared with the strictly linear models, CLMS and SS1, is highlighted. Conforming with the analysis, the strictly linear algorithms, CLMS and SS1, yielded inaccurate and oscillating frequency estimates, while the widely linear algorithms, ACLMS and SS2, yielded accurate estimates. In both simulations, the state space based widely linear Kalman filter based algorithm, SS2, had a faster convergence rate and lower steady state error than the stochastic gradient based widely linear ACLMS algorithm.

V. Conclusions

We have shown that by utilising widely linear modeling, the system frequency in three phase power systems can be

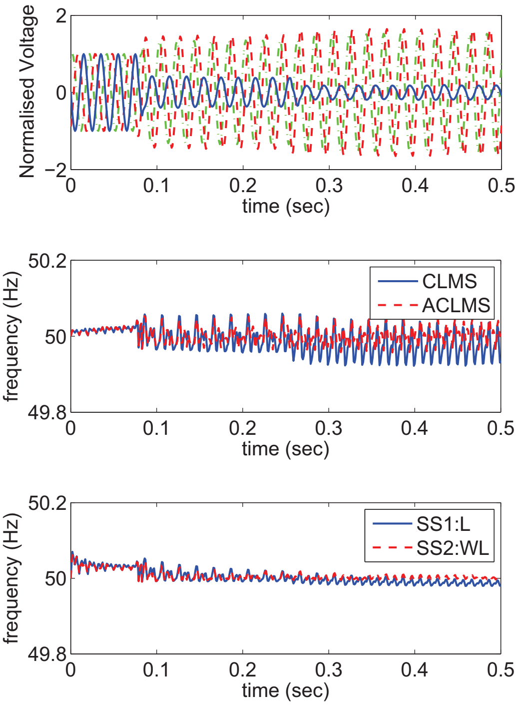

Fig. 4. Frequency estimation for real-world unbalanced three-phase voltages, where an initially balanced system experienced a single-line short with earth.

accurately estimated, particularly for unbalanced systems since the system becomes noncircular when unbalanced. Frequency estimation has been addressed from a state space perspective, and the superiority of the widely linear (augmented) complex Kalman filter over the stochastic gradient based widely linear (augmented) complex LMS algorithm has been illustrated on real world examples.

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