Compressed Sensing: Challenges and Emerging Topics

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Compressed sensing

Engineering Challenges in CS:

- What is the right signal model?

  Sometimes obvious, sometimes not. When can we exploit additional structure?

- How can/should we sample?

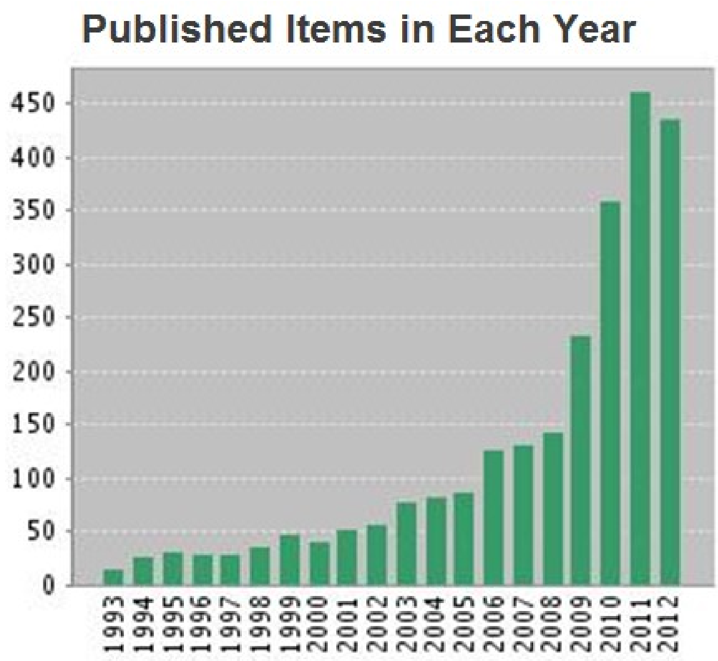
  Physical constraints; can we sample randomly; effects of noise; exploiting structure; how many measurements?

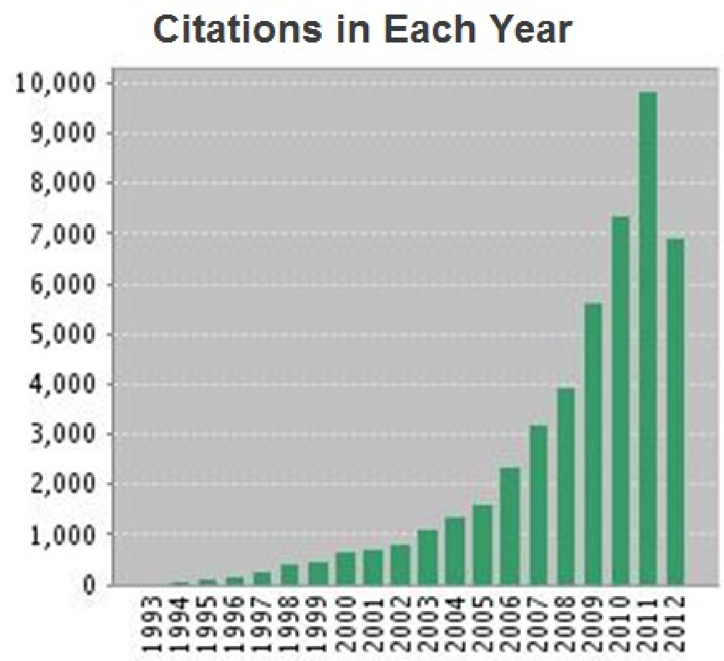
- What are our application goals?

  Reconstruction? Detection? Estimation?

CS today – the hype!

Papers published in Sparse Representations and CS [Elad 2012]





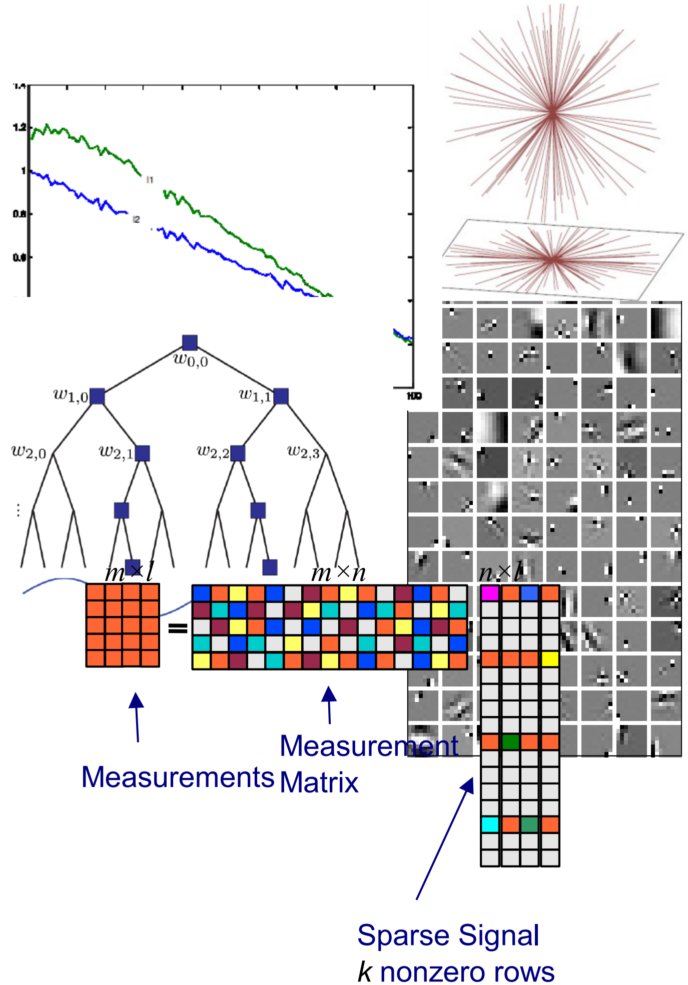
Lots of papers….. lots of excitement…. lots of hype….

Google About 4,500,000 results **(0.37 seconds)**

CS today: - new directions & challenges

There are many new emerging directions in CS and many challenges that have to be tackled.

- Fundamental limits in CS
- Structured sensing matrices
- Advanced signal models
- Data driven dictionaries
- Effects of quantization
- Continuous (off the grid) CS
- Computationally efficient solutions
- Compressive signal processing



Compressibility and Noise Robustness

Noise Robustness

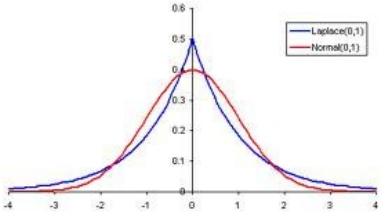
CS is robust to measurement noise:

RIP $\Longrightarrow$ $\|\Delta(\Phi x + \epsilon) - x\|_2 \leq C_1 \sigma_k(x)_1 + C_2 \|\epsilon\|_2$

What about signal errors: $\Phi(x + e) = y$?

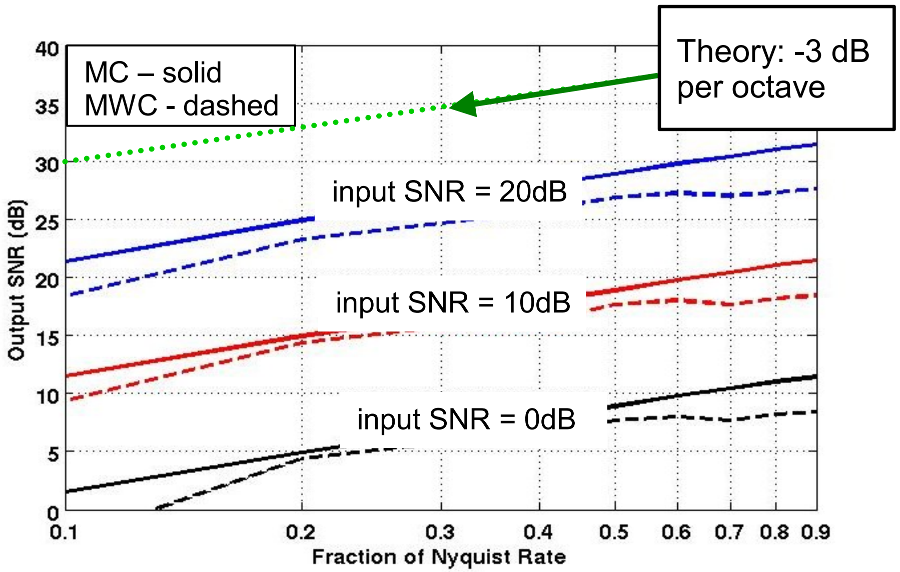
No free lunch!

- Detecting signals through wide band receiver noise: noise folding!
  - 3dB SNR loss per 2 undersampling [Treichler, et al 2011]
- Imaging: “tail folding” of coefficient distribution
  - Fundamental limits in terms of the compressibility of the pdf [D. and Guo. 2011, Gribonval et al 2012]



SNR/SDR vs Undersampling

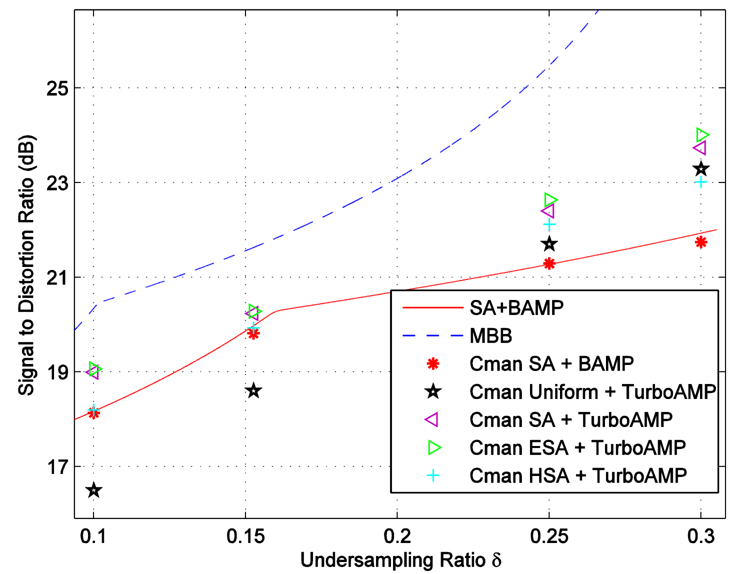
Wideband spectral sensing



Adaptive sensing can retrieve lost SNR [Haupt et al 2011]

Trade-off – better dynamic range for SNR loss [Treichler, et al 2011]

Bounds guide sampling strategies and provide fundamental limits to CS imaging performance



Compressible distributions

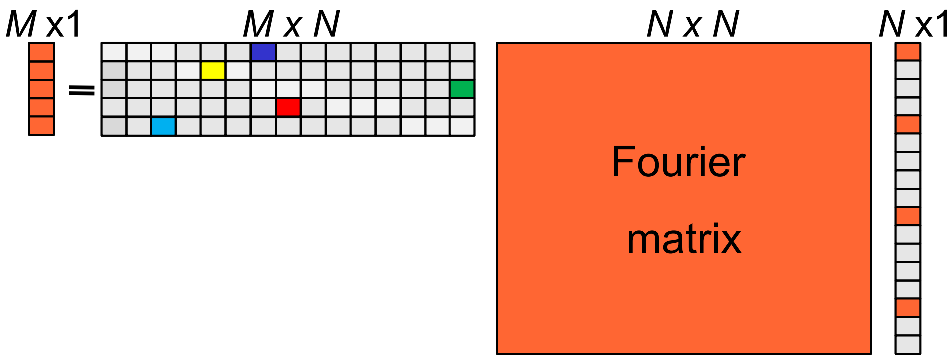
Sensing matrices

Structured CS sensing matrices

i.i.d. sensing matrices are really only of academic interest.

Need to consider wider classes, e.g.:

- Random rows of DFT [Rudelson & Vershynin 2008]



δ-RIP of order k with high probability if:

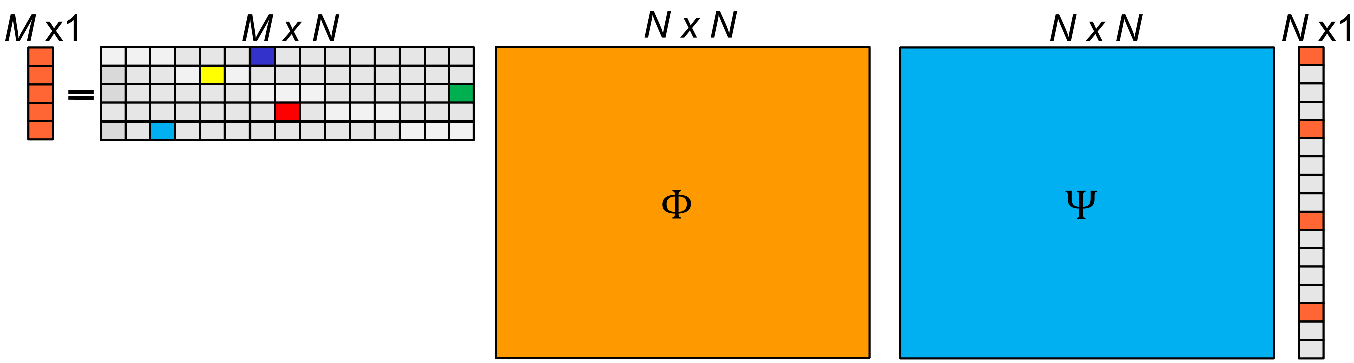
$$m \sim \mathcal{O}(k\, \delta^{-2} \log^4 N)$$

Structured CS sensing matrices

i.i.d. sensing matrices are really only of academic interest.

Need to consider wider classes, e.g.:

- Random samples of a bounded orthogonal system [Rauhut 2010]



Also extends to continuous domain signals.

δ-RIP of order k with high probability if:

$$m \sim \mathcal{O}(k\,\mu(\Phi,\Psi)^2 \delta^{-2} \log^4 N)$$

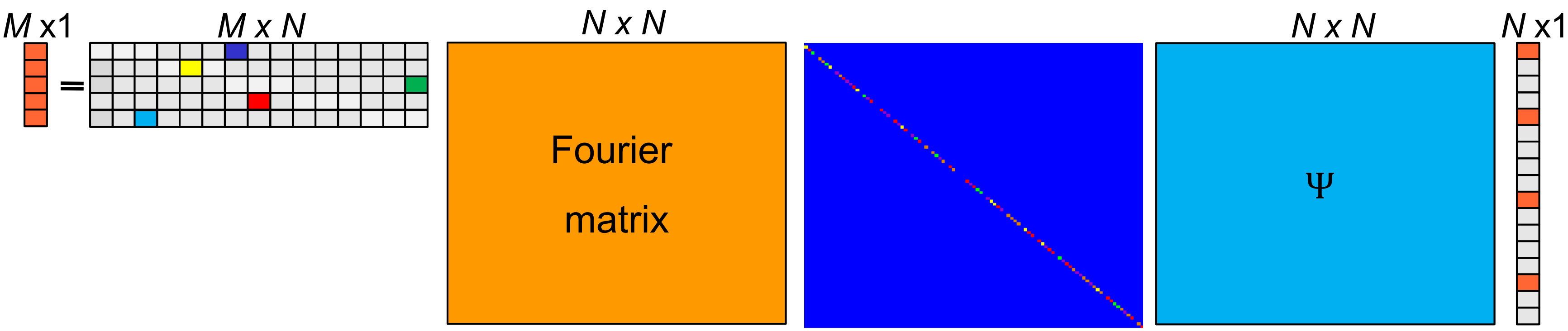
where $\mu(\Phi,\Psi) = \max_{1\le i<j\le N} \left|\langle \Phi_i, \Psi_j \rangle\right|$ is called the mutual coherence

Structured CS sensing matrices

i.i.d. sensing matrices are really only of academic interest.

Need to consider wider classes, e.g.:

- Universal Spread Spectrum sensing [Puy et al 2012]



Sensing matrix is random modulation followed by partial Fourier matrix. δ-RIP of order k with high probability if:

$$m \sim \mathcal{O}(k\,\delta^{-2}\log^5 N)$$

Independent of basis Ψ!

Generalized Dimension Reduction

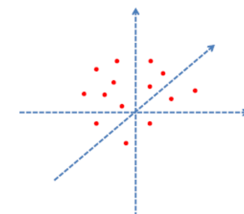
Compressed sensing matrices can be used to preserve information beyond sparsity. Robust embeddings (RIP for difference vectors):

$$(1-\delta)\|x-x'\|_2 \leq \|\Phi(x-x')\|_2 \leq (1+\delta)\|x-x'\|_2$$

hold for many low dimensional sets.

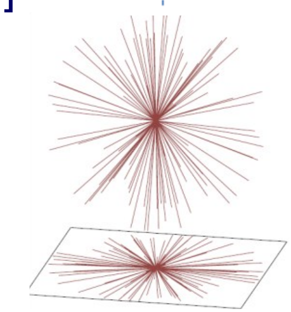
- Sets of n points [Johnston and Lindenstrauss 1984]

$$m \sim \mathcal{O}(\delta^{-2}\log n)$$



- Arbitrary Union of L k-dimensional subspaces [Blumensath and D. 2009]

$$m \sim \mathcal{O}(\delta^{-2}(k+\log L))$$

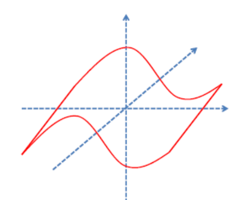


- Set of r-rank $\mathrm{n} \times l$ matrices [Recht et al 2010]

$$m \sim \mathcal{O}(\delta^{-2} r(n+l)\log nl)$$

- d-dimensional affine subspaces [Sarlos 2006]

$$m \sim \mathcal{O}(\delta^{-2} d)$$



- d-dimensional manifolds [Baraniuk and Wakin 2006, Clarkson 2008]

$$m \sim \mathcal{O}(\delta^{-2} d)$$

Advanced signal models & algorithms

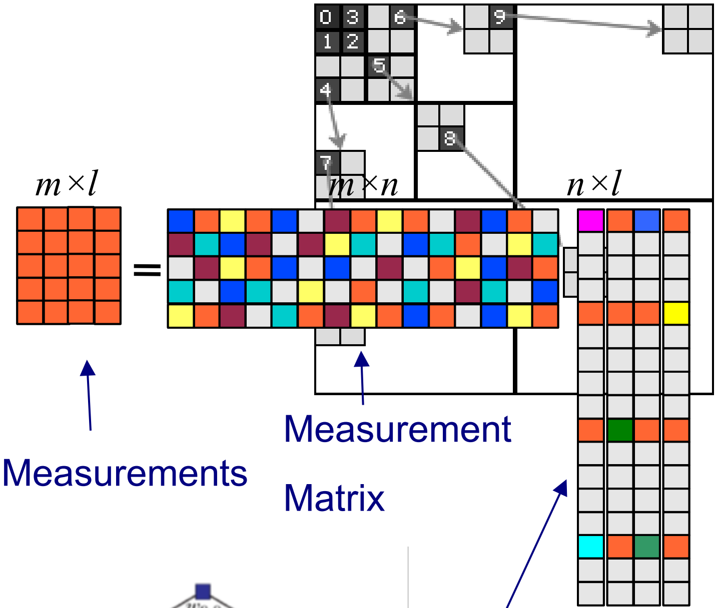
CS with Low Dimensional Models

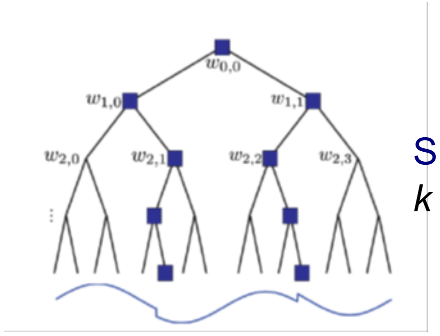
What about sensing with other low dimensional signal models?

- Matrix completion/rank minimization
- Phase retrieval
- Tree based sparse recovery
- Group/Joint Sparse recovery
- Manifold recovery

… towards a general model-based CS?

[Baraniuk et al 2010, Blumensath 2011]





Matrix Completion/Rank minimization

Retrieve the unknown matrix $X \in \mathbb{R}^{N \times L}$ from a set of linear observations

$$y = \Phi(X),\ y \in \mathbb{R}^m \text{ with } m < NL.$$

Suppose that X is rank r.

Relax!

as with L_1 min., we convexify: replace $\mathrm{rank}(X)$ with the nuclear norm $\|X\|_* = \sum_i \sigma_i$, where σ_i are the singular values of X.

$$\hat{X} = \underset{X}{\mathrm{argmin}} \|X\|_* \text{ subject to } \Phi(X) = y$$

Random measurements (RIP) $\longrightarrow$ successful recovery if

$$m \sim \mathcal{O}(r(N+L)\log NL)$$

e.g. the Netflix prize

– rate movies for individual viewers.

Phase retrieval

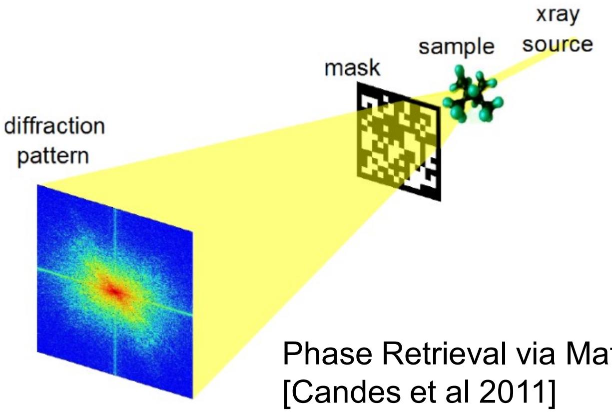
Generic problem:

Unknown $x \in \mathbb{C}^n$,

magnitude only observations: $y_i = |\mathrm{A_i} x|^2$

Applications

- X-ray crystallography
- Diffraction imaging
- Spectrogram inversion



Phase Retrieval via Matrix Completion [Candes et al 2011]

Phaselift

Lift quadratic ⟶ linear problem using rank-1 matrix $X = xx^H$

Solve: $\hat{X} = \underset{X}{\operatorname{argmin}} \|X\|_*$ subject to $\mathcal{A}(X) = y$

Provable performance but lifting space is huge!

… surely more efficient solutions?

Tree Structured Sparse Representations

Sparse signal models are type of "union of subspaces" model [Lu & Do 2008, Blumensath & Davies 2009] with an exponential number of subspaces.

$$\#\text{ subspaces} \approx \left(\frac{N}{k}\right)^k \quad \text{(Stirling approx.)}$$

Tree structure sparse sets have far fewer subspaces

$$\#\text{ subspaces} \approx \frac{(2e)^k}{k+1} \quad \text{(Catalan numbers)}$$

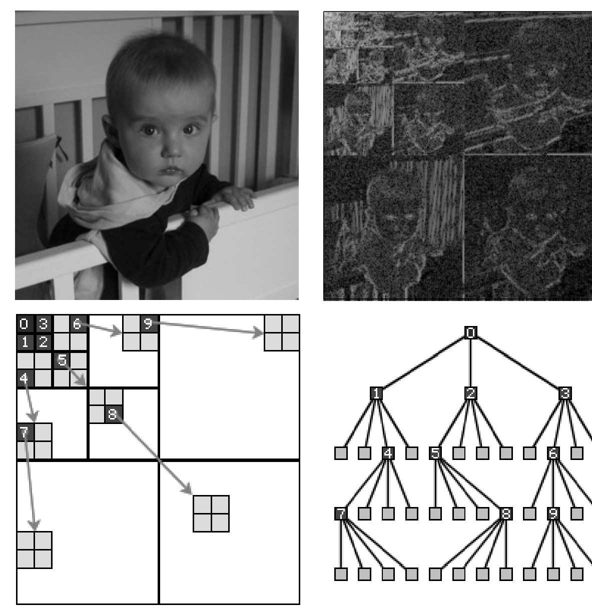
Example exploiting wavelet tree structures

Classical compressed sensing: stable inverses exist when

$$m \sim \mathcal{O}(k \log(N/k))$$

With tree-structured sparsity we only need [Blumensath & D. 2009]

$$m \sim \mathcal{O}(k)$$



Algorithms for model-based recovery

Baraniuk et al. [2010] adapted CoSaMP & IHT to construct provably good model-based recovery algorithms.

original

sparse reconstruction

Tree sparse reconstruction

Blumensath [2011] adapted IHT to reconstruct <u>any</u> low dimensional model from RIP-based CS measurements:

$$x^{n+1} = \mathcal{P}_{\mathcal{A}}\left(x^n + \mu\Phi^{\mathrm{T}}(\mathrm{y} - \Phi x^n)\right)$$

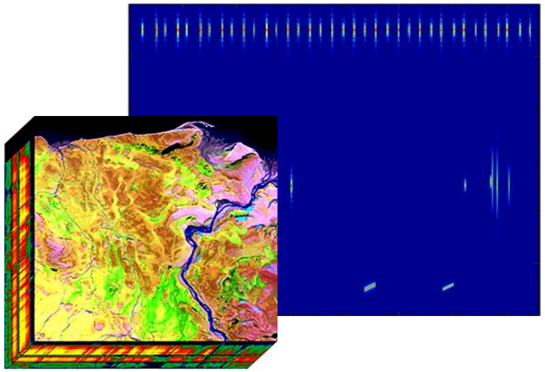
where $\mu \sim N/m$ is the step size, $\mathcal{P}_{\mathcal{A}}$ is the projection onto the signal model.

Requires a computationally efficient $\mathcal{P}_{\mathcal{A}}$ operator.

Sparse Multiple Measurement Vector Problem

Find a row sparse matrix $X \in \mathbb{R}^{N\times l}$ given the multiple measurements $Y \in \mathbb{R}^{m\times l}$. Applications such as:

- multiband spectral sensing,
- hyperspectral imaging

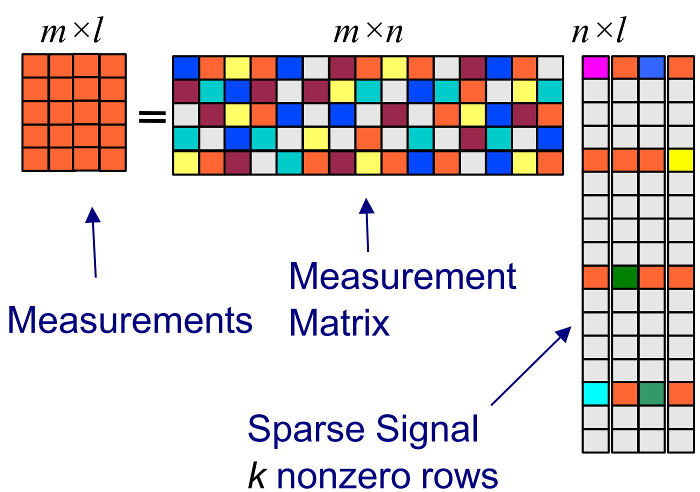


L_0 solution:

$$\hat{X} = \underset{X}{\operatorname{argmin}}\ |\mathrm{row\ supp}(X)|\ \ s.t.\ \Phi X = Y$$

Difficulty of inverse problem depends on rank of Y [Eldar and D. 2013]

When $\mathrm{rank}(Y) = k$ the problem can be solved with the MUSIC algorithm.

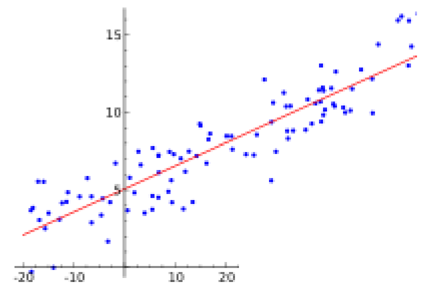


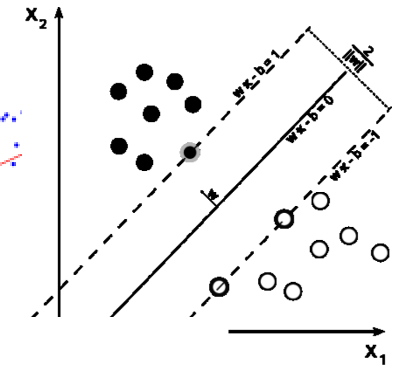
Compressed Signal Processing

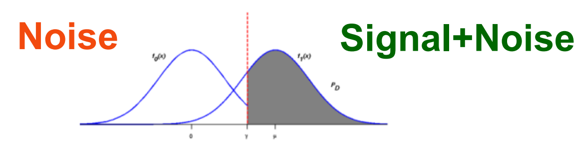
Compressed Signal Processing

There is more to life than signal reconstruction:

- Detection
- Classification
- Estimation
- Source separation







$$\mathcal{H}_0 : y = \Phi n$$

$$\mathcal{H}_1 : y = \Phi(s + n)$$

May not wish to work in large ambient signal space, e.g. ARGUS-IS Gigapixel camera

CS measurements can be information preserving (RIP)… offers the possibility to do all your DSP in the compressed domain!

Without reconstruction what replaces Nyquist?

Compressive Detection

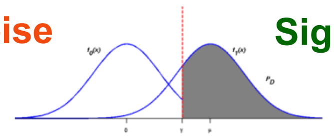
The ~~Matched~~ Smashed Filter [Davenport et al 2007]

Detection can be posed as the following hypothesis test:

$$\mathcal{H}_0 : z = hn$$

$$\mathcal{H}_1 : z = h(s + n)$$

Noise



Signal+Noise

The optimal (in Gaussian noise) matched filter is $h = s^H$

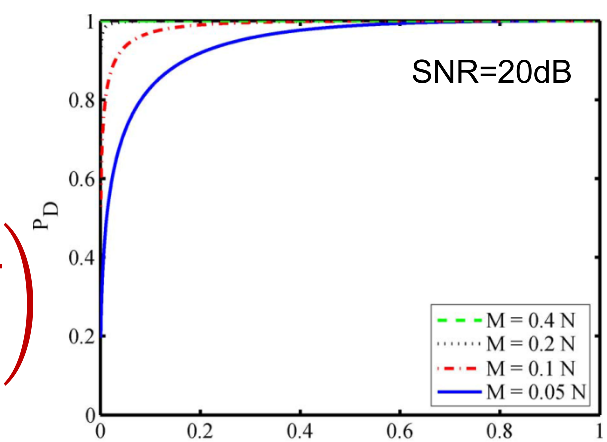
Given CS measurements: $y = \Phi s$, the matched filter (applied to y) is:

$$h = s^H \Phi (\Phi \Phi^H)^{-1}$$

Then

$$P_D \approx Q\left(Q^{-1}(\alpha) - \sqrt{\frac{m}{N}}\sqrt{SNR}\right)$$

Q - the Q-function, α – Prob. false alarm rate



[Davenport et al 2010]

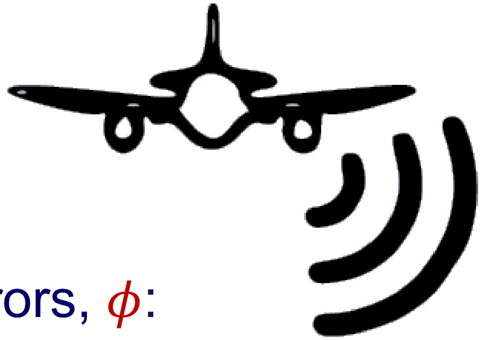
Joint Recovery and Calibration

Estimation and recovery, e.g. on-line calibration.

Compressed Calibration

Real Systems often have unknown parameters θ that need to be estimated as part of signal reconstruction.

$$y = \Phi(\theta)x$$



Can we simultaneously estimate x and θ?

Example – Autofocus in SAR

Imperfect estimation of scene centre leads to phase errors, ϕ:

$$Y = \mathrm{diag}\left(e^{j\phi}\right)h(X)$$

X- scene reflectivity matrix, Y- observed phase histories, $h(\cdot)$- sensing operator.

Uniqueness conditions from dictionary learning theory [Kelly et al. 2012].

Joint Recovery and Calibration

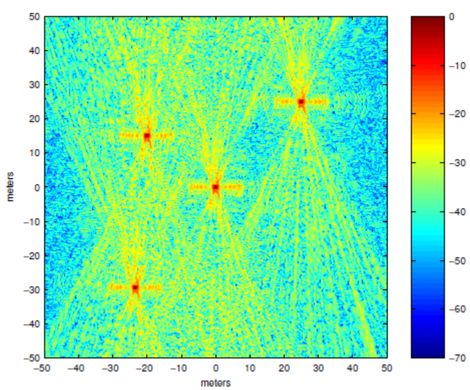
Compressed Autofocus:

Perform joint estimation and reconstruction:

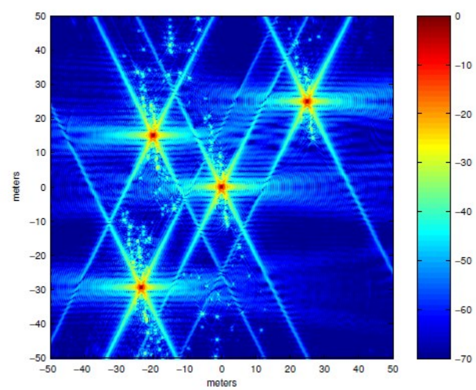
$$\min_{X,d} \|X\|_1 \quad \text{subject to } \|Y - \mathrm{diag}(d)h(X)\|_F \leq \epsilon$$

$$\text{and} \quad d_i d_i^* = 1, i = 1, \ldots, N$$

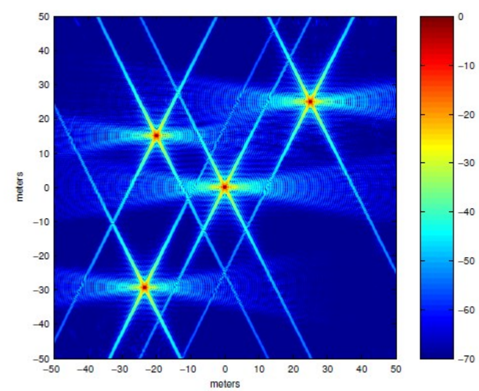
- Fast alternating optimization schemes available
- Provable performance? **Open**



No phase correction



Post-recon. autofocus



Compressive autofocus

Summary

Compressive Sensing (CS)

- combines sensing, compression, processing
- exploits low dimensional signal models and incoherent sensing strategies

Still lots to do…

- Developing new and better model-based CS algorithms and acquisition systems
- New emerging field of compressive signal processing
- Exploit dimension reduction in computation: randomized linear algebra,… big data!

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