Foundations of Compressed Sensing

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Part I: Foundations of CS

- Introduction to sparse representations & compression
- Compressed sensing – motivation and concept
- Information preserving sensing matrices
- Practical sparse reconstruction
- Summary & engineering challenges

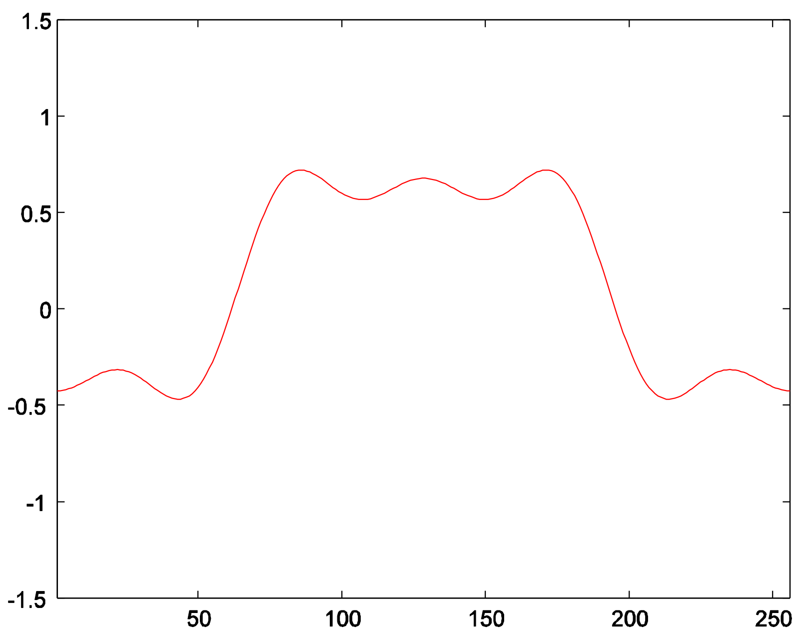
Sparse representations and compression

Fourier Representations

The Frequency viewpoint (Fourier, 1822):

Joseph Fourier

Signals can be built from the sum of harmonic functions (sine waves)



Atomic representation:

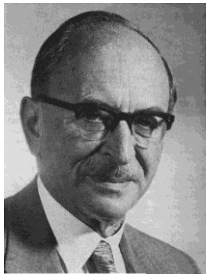
$$x(t) = \sum_k c_k e^{\omega_0 k t} = F\boldsymbol{c}$$

signal

Fourier coefficients

Harmonic functions

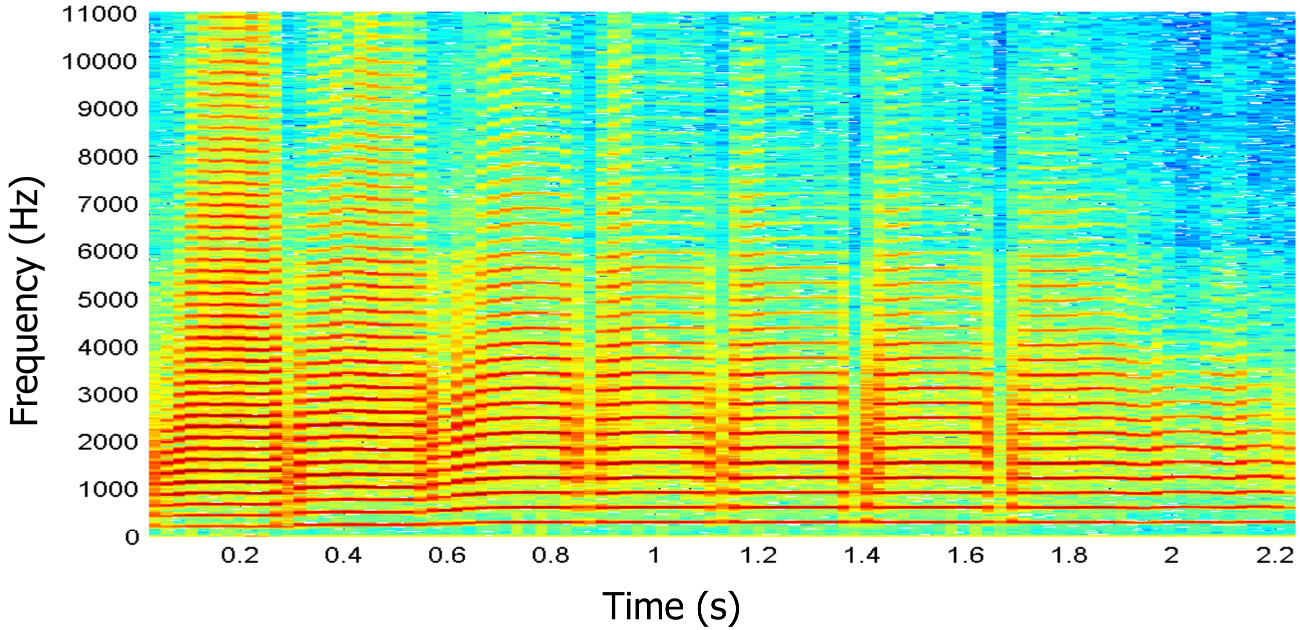
Time-Frequency representations



Time <u>and</u> Frequency (Gabor)

"Theory of Communication," J. IEE (London) , 1946

"… a new method of analysing signals is presented in which time and frequency play symmetrical parts…"



Atomic (dictionary) representation:

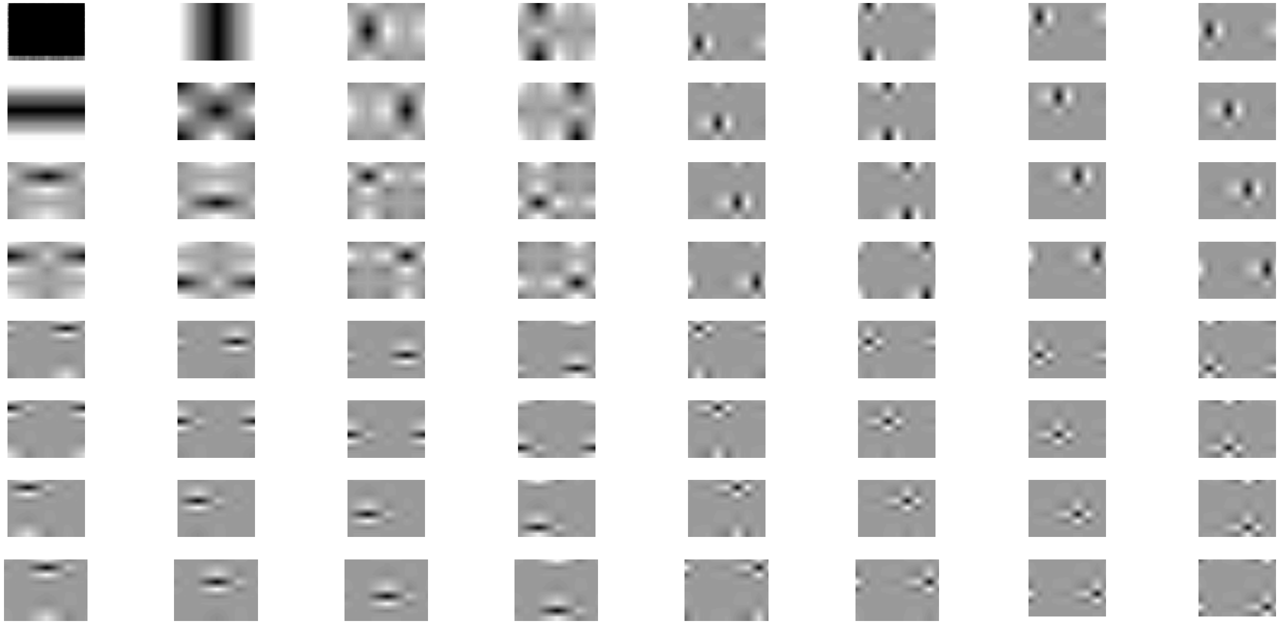
$$x(t) = \sum_n \sum_k c_{n,k} \times g(t - n\tau)e^{j\omega kn} = \Phi \boldsymbol{c}$$

Space-Scale representations

the wavelet viewpoint:

***"Daubechies, Ten Lectures on Wavelets,"* SIAM 1992**

Images can be built of sums of *wavelets*. These are multi-resolution edge-like (image) functions.



and many other representations

… more recently:

chirplets,

curvelets,

edgelets,

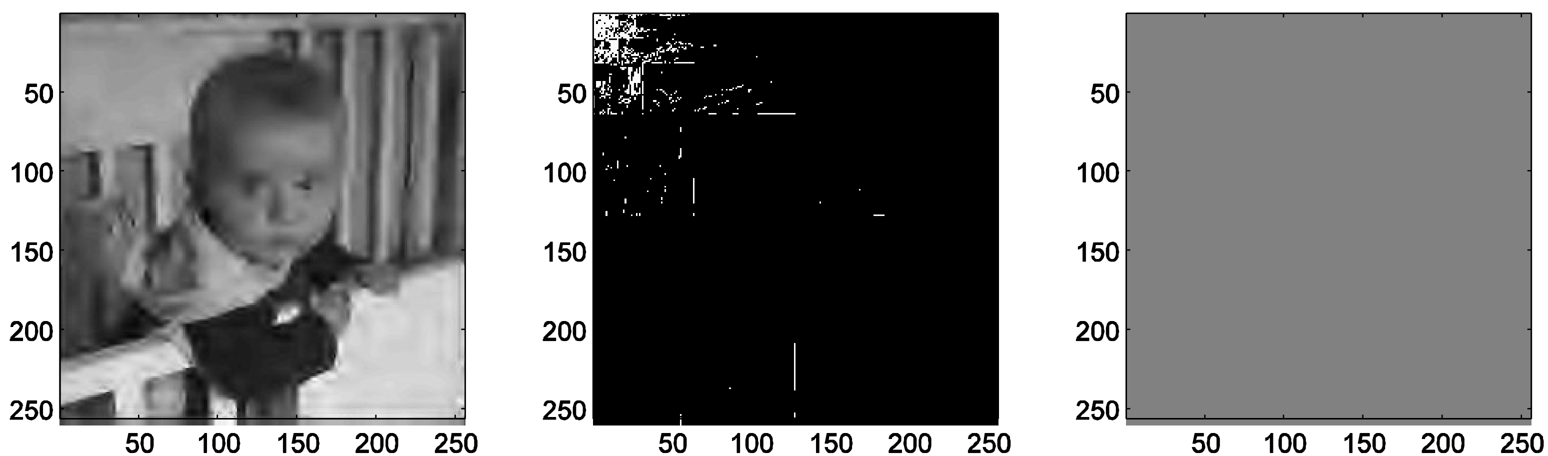
wedgelets, …

dictionary learning...

Coding signals of interest

What is the difference between quantizing a signal/image in the transform domain rather than the signal domain?

Compressed to 0.1 bits per pixel



Quantization in wavelet domain

Tom's nonzero wavelet coefficients

Quantization in pixel domain

Good representations are efficient – e.g. sparse!

Sparsity & Compression

A vector x is k-sparse, if only k of its elements are non-zero.

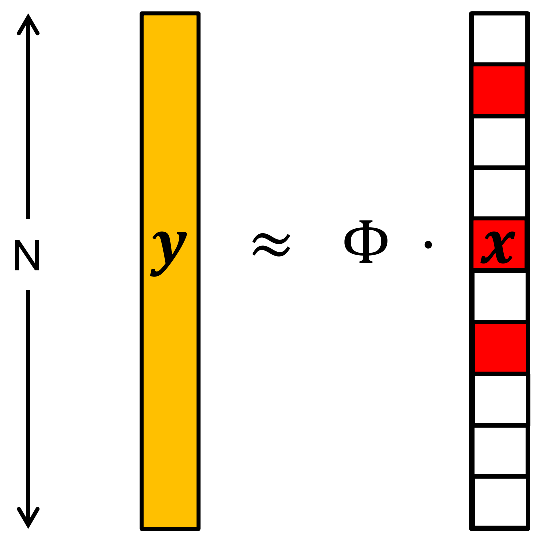
$$[0\ 0.5\ 0\ 0\ 0.1\ 0\ -0.2\ 0\ 0\ 0\ 0]^T$$

Such vectors have only k-degrees of freedom (k-dimensional) and there are "N choose k", $\binom{N}{k}$, possible combinations of nonzero coefficients.

Coding cost:

N floats

$= \mathcal{O}(N)$ bits



Coding cost:

k floats + $\log_2 \binom{N}{k}$ bits

$= \mathcal{O}\big(k \log_2(N/k)\big)$ bits

$$\boldsymbol{y} \approx \Phi \boldsymbol{x}$$

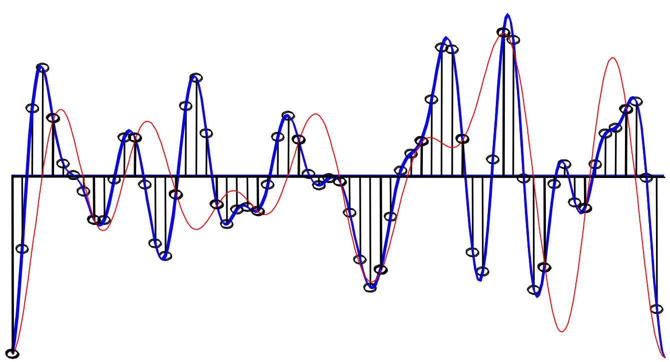
Compressed sensing: motivation and concepts

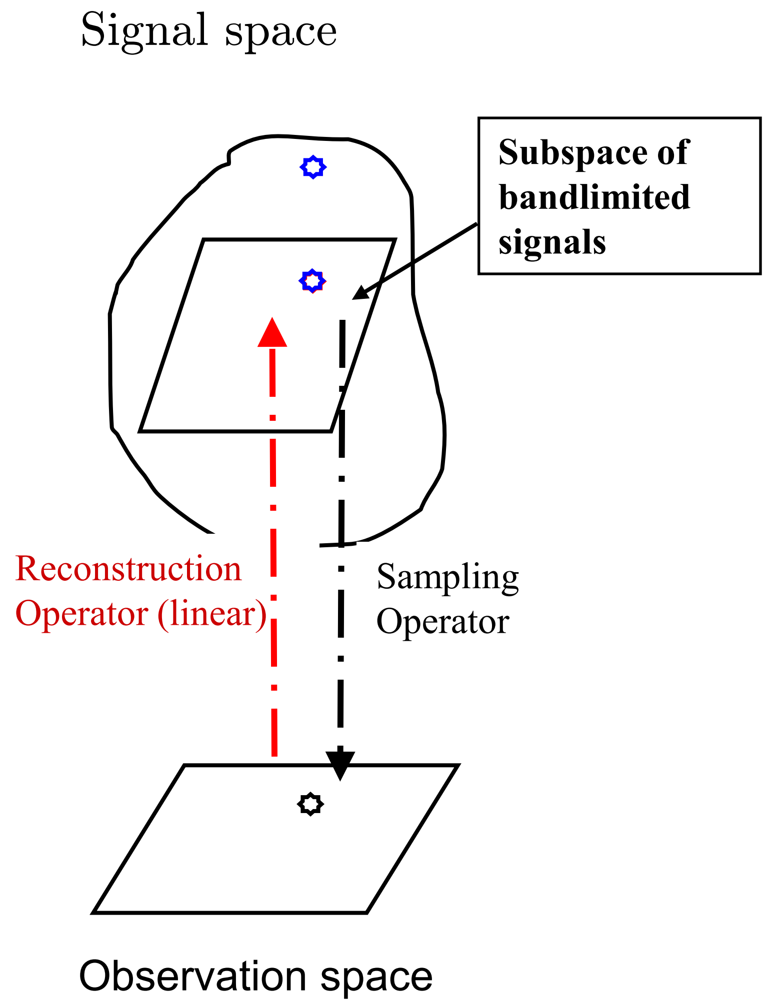
Classical Sampling Theory

The **Whittaker–Kotelnikov–Shannon Sampling Theorem** states:

"Exact reconstruction of a continuous-time signal from discrete samples is possible if the signal is bandlimited and the sampling frequency is greater than twice the highest frequency."

Sampling below this rate introduces aliasing

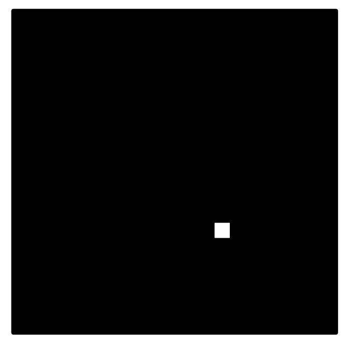




Generalized Sampling

Different ways to measure…

Equivalent to inner product with various functions

$\Big\langle$  ,  $\Big\rangle$

pointwise sampling, tomography, coded aperture,…

Generalized Sampling

Different ways to measure…

Equivalent to inner product with various functions

pointwise sampling, tomography, coded aperture,…

Generalized Sampling

Different ways to measure…

Equivalent to inner product with various functions

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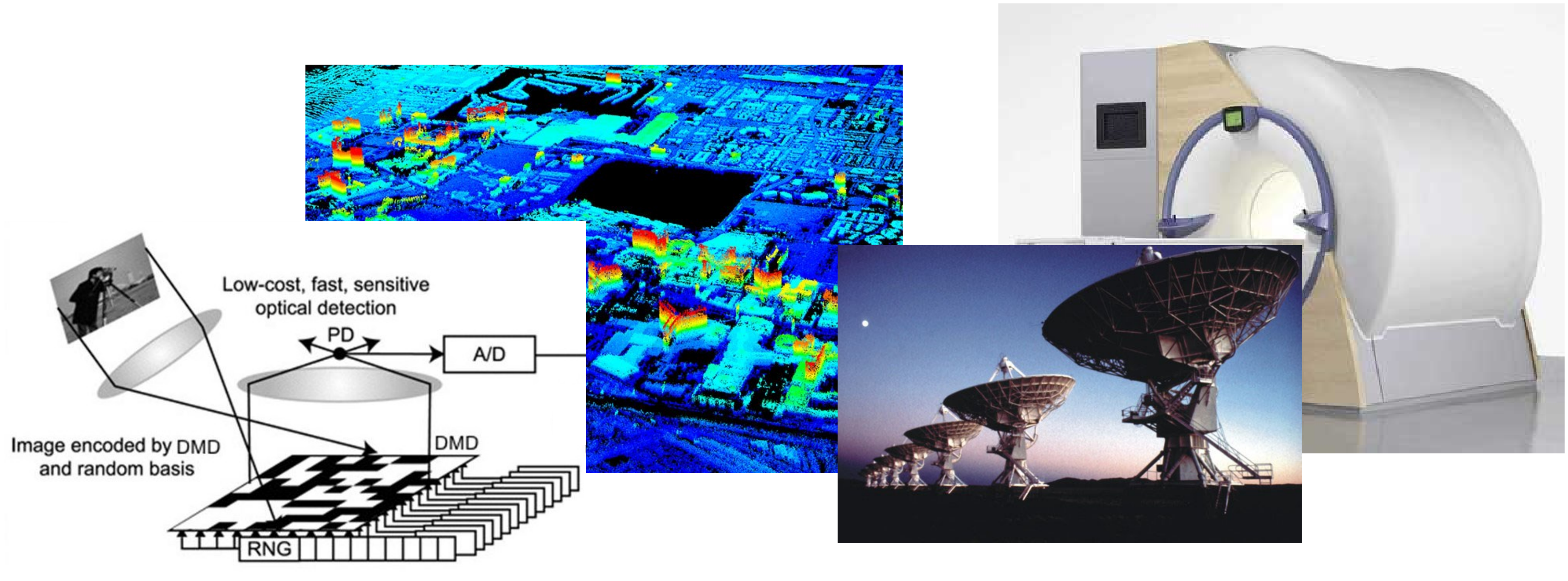
pointwise sampling, tomography, coded aperture,…

New Challenges

Challenge #1: Insufficient Measurements

Complete measurements can be costly, time consuming and sometimes just impossible!



New Challenges

Challenge #2: Too much data

e.g.

DARPA ARGUS-IS

1.8 Gpixel image sensor

15cm resolution, 12 frames a second

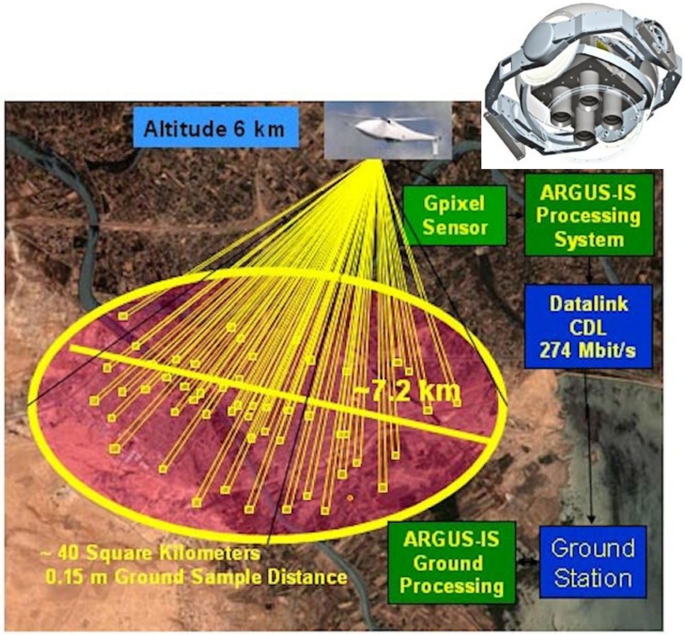
Giving a video rate output:

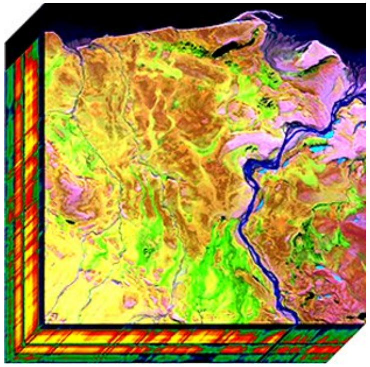
444 Gbits/s

… but the comms link data rate is:

274 Mbits/s

Currently visible spectrum. What about hyperspectral?…





The new hope: Compressed Sensing

E. Candès, J. Romberg, and T. Tao, **"Robust Uncertainty principles: Exact signal reconstruction from highly incomplete frequency information,"** IEEE Trans. Information Theory, 2006

D. Donoho, **"Compressed sensing,"** IEEE Trans. Information Theory, 2006

Why can't we just sample signals at the "Information Rate"?

When compressing a signal we typically take lots of samples (sampling theorem), move to a transform domain, and then throw most of the coefficients away! Can we just sample what we need?

Yes! …and more surprisingly we can do this non-adaptively.

Compressed sensing Overview

Observe $\boldsymbol{x} \in \mathbb{R}^N$ via $m \ll N$ measurements, $\boldsymbol{y} \in \mathbb{R}^m$ where $\boldsymbol{y} = \Phi \boldsymbol{x}$

Compressed Sensing assumes a compressible set of signals, i.e. approximately k-sparse.

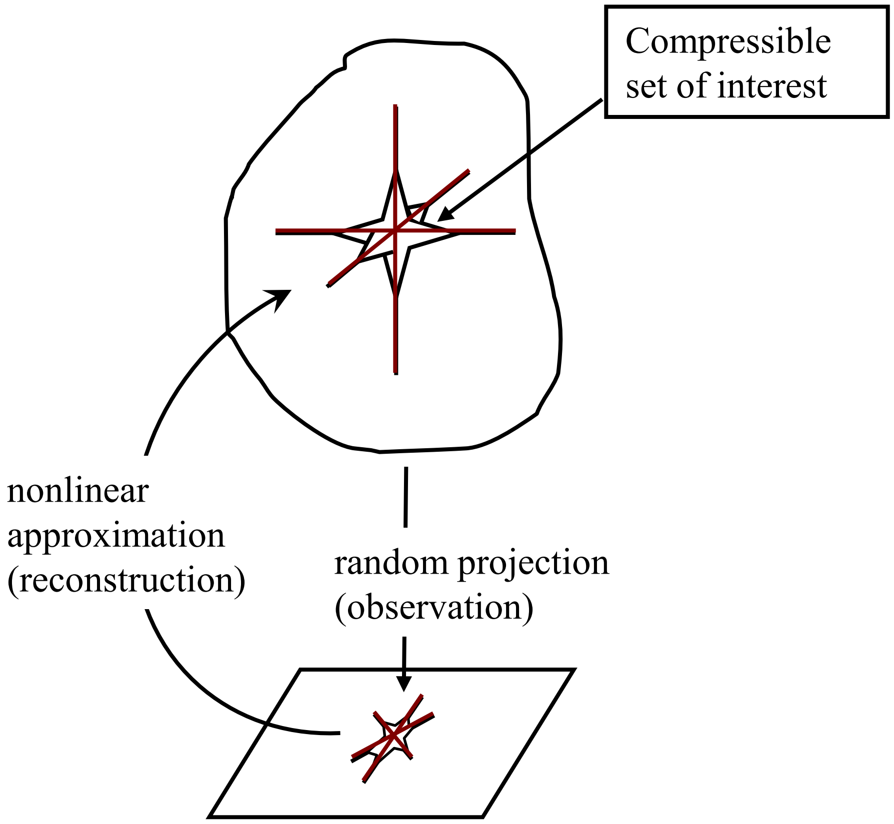
Using approximately

$$m \geq \mathcal{O}\left(k \log_2 \frac{N}{k}\right)$$

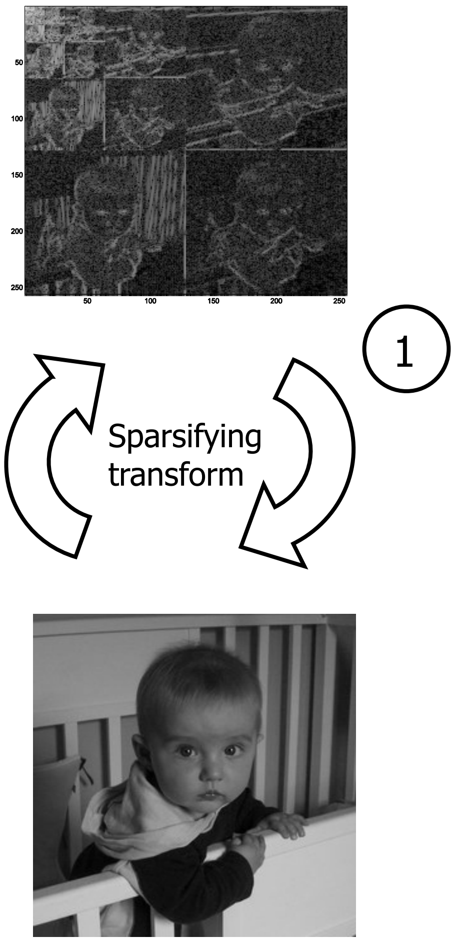
random projections for measurements we have little or no information loss.

Signal reconstruction by a nonlinear mapping.

Many practical algorithms with guaranteed performance e.g. L_1 min., OMP, CoSaMP, IHT.

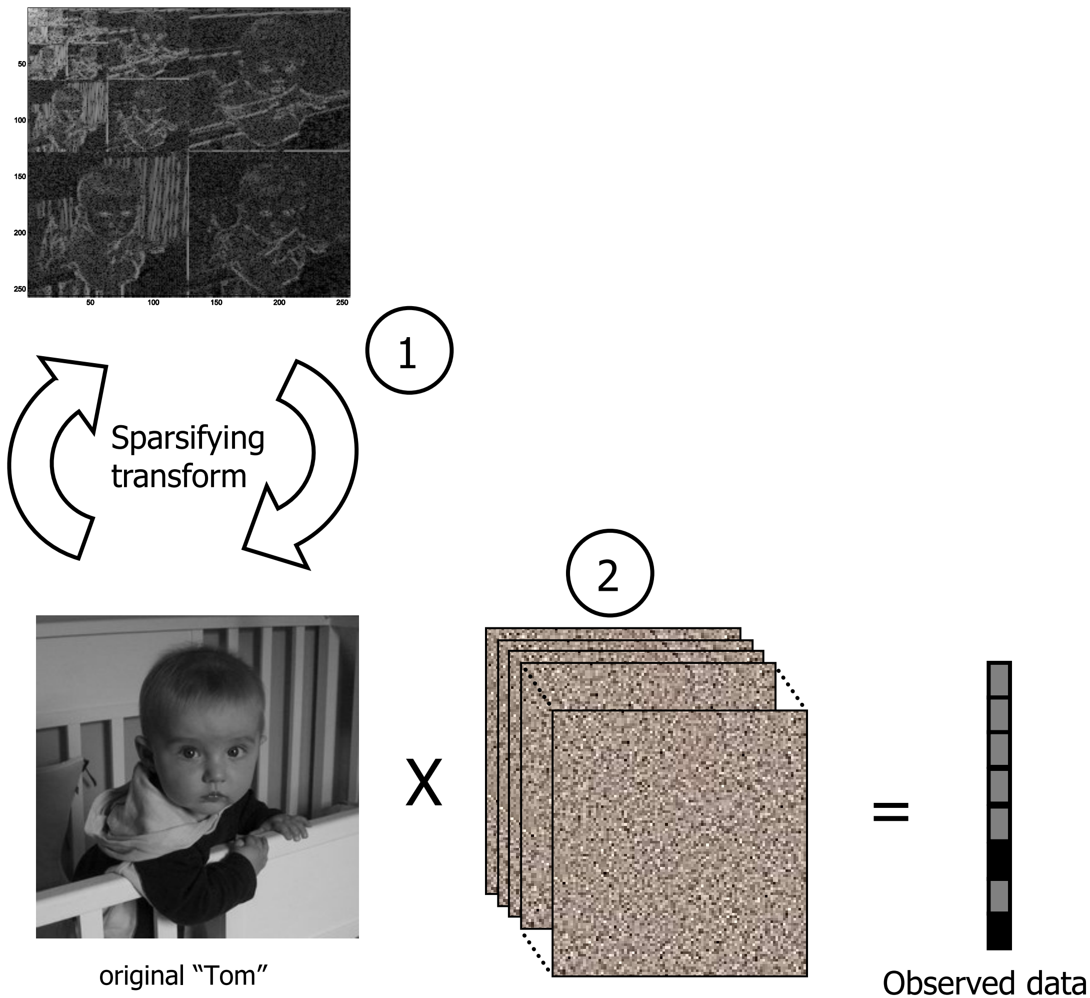


CS acquisition/reconstruction principle



original “Tom”

CS acquisition/reconstruction principle



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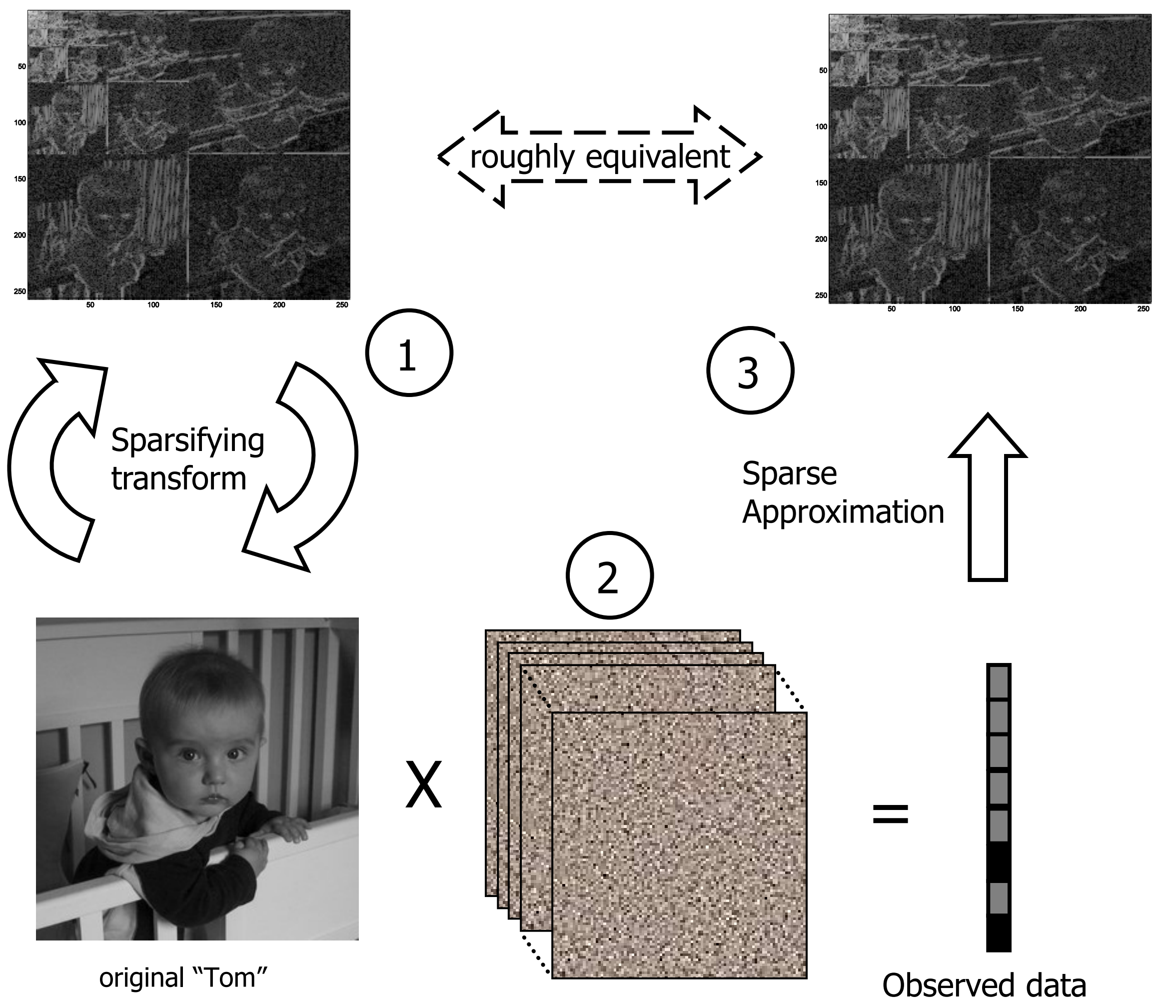

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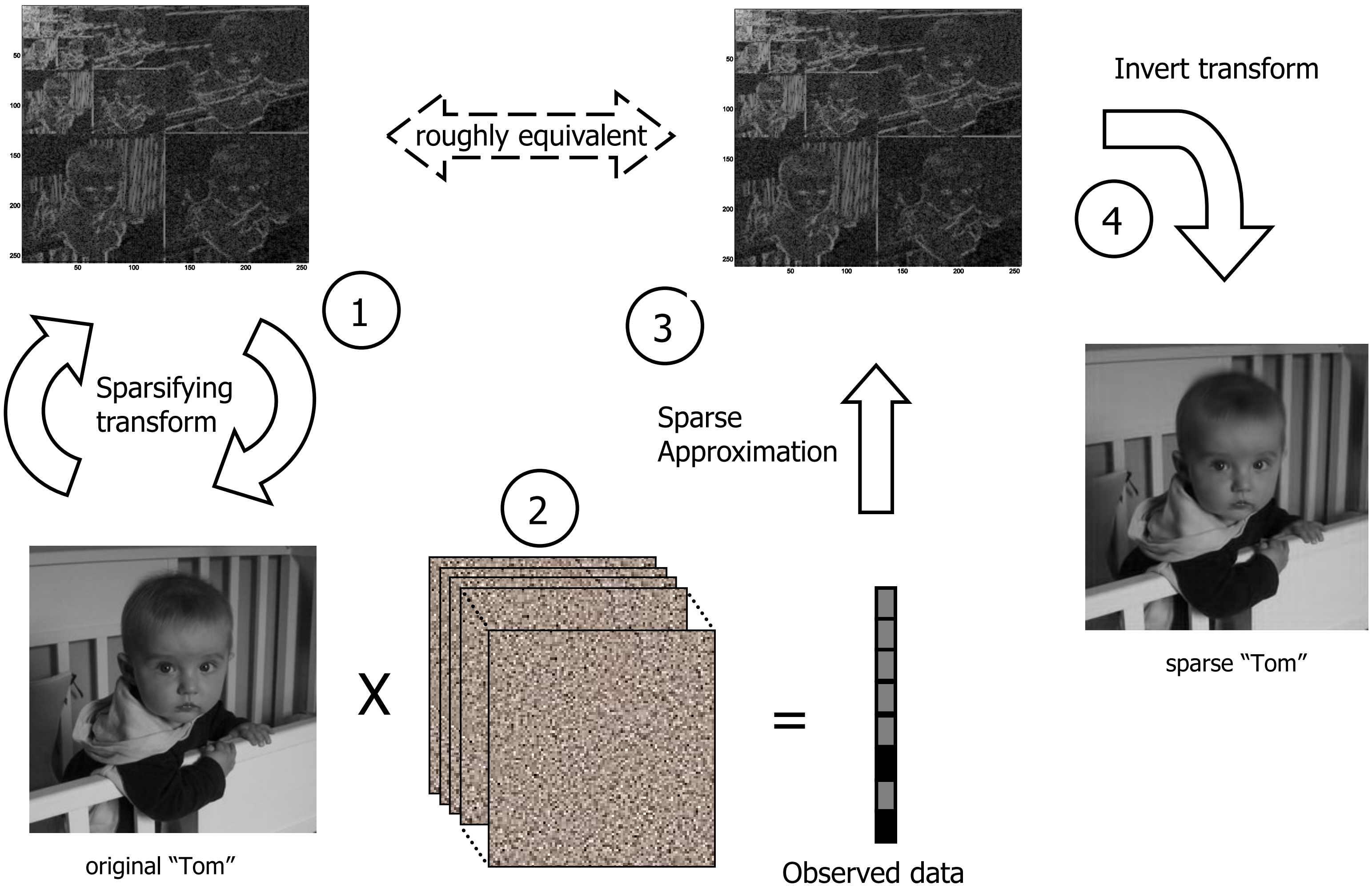
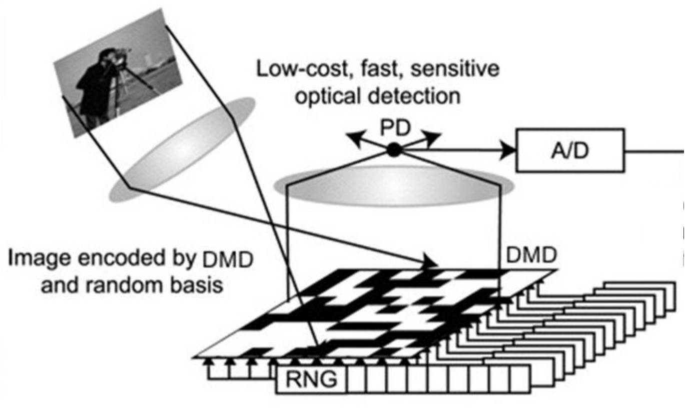


Potential applications

Compressed Sensing provides a new way of thinking about signal acquisition.

Applications areas include:

- Medical imaging
- Hyperspectral imaging
- Astronomical imaging
- Distributed sensing
- Radar sensing
- Geophysical (seismic) exploration
- High rate A/D conversion
  (DARPA A2I research program)



Rice University single pixel camera

Information preserving sensing matrices

Information preservation

Underdetermined ($m < n$) linear systems are not invertible: $\Phi x = \Phi x' \nRightarrow x = x'$.

However, they may be invertible restricted to the sparse set.

Define the null space of Φ as: $\mathcal{N}(\Phi) = \{z : \Phi z = 0\}$.

Then

$$\Phi x = \Phi x' \implies x = x'$$

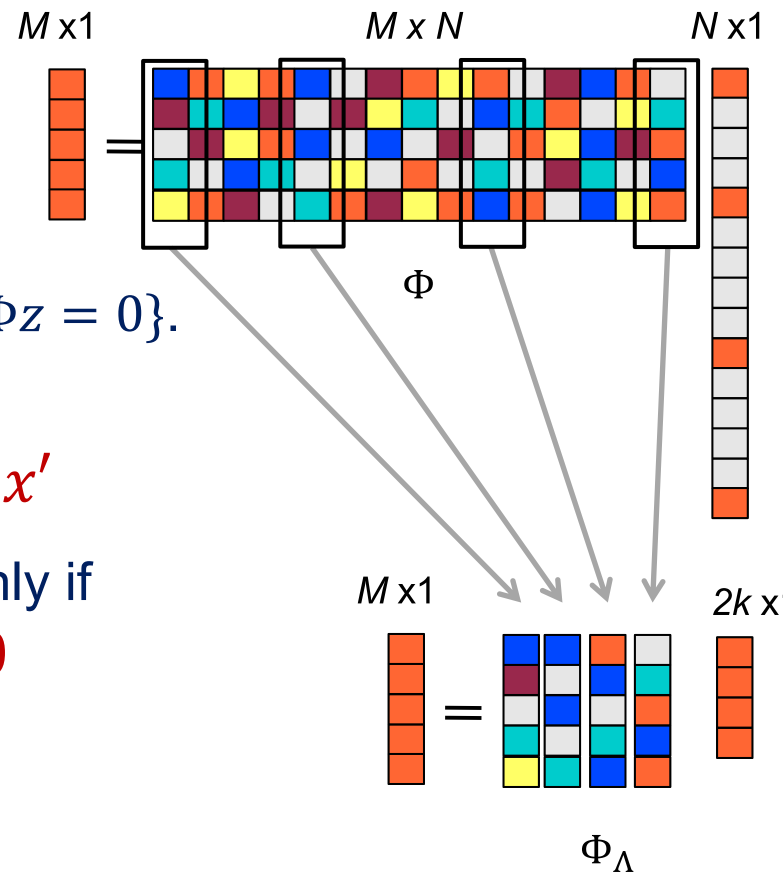
for any k-sparse vectors, x and $x,'$ if and only if

$$\Phi z = 0 \implies z = 0$$

for all 2k-sparse vectors, z.

That is:

1. the null space of Φ cannot contain 2k-sparse vectors, or
2. submatrices, Φ_Λ, with column index sets $|\Lambda| = 2k$ must be full rank.



Uniqueness of the inverse map

For almost every $\Phi: \mathbb{R}^N \to \mathbb{R}^m, k \leq m/2$

$$\Phi x_1 \neq \Phi x_2 \ \forall \text{ k−sparse } x_1 \neq x_2$$

i.e. we can retrieve the original k-sparse vector using the following L_0 minimization scheme:

$$\hat{x} = \underset{x}{\operatorname{argmin}} \ \|x\|_0 \text{ subject to } \Phi x = y$$

Where $\|x\|_0$ is the L_0 (quasi-) norm that counts the number of nonzero elements in x. Unfortunately L_0 solution may not be robust and solving the L_0 minimization is known to be NP complete (computationally infeasible).

Robust Null Space Properties

In order to achieve robustness we need to consider stronger NSPs

[Cohen et al. 2009] introduced the notion of Instance Optimality and showed that the following are equivalent up to a change in constant C

1. There existing a reconstruction mapping, Δ, such that for all x:

$$\|\Delta(\Phi x) - x\|_1 \leq C\sigma_k(x)_1$$

   where $\sigma_k(x)_1$ is the L_1 best k-term approximation error of x

2. Φ satisfies the following NSP:

$$\|z_\Lambda\|_1 \leq C'\sigma_{2k}(z)_1$$

   for all $z \in \mathcal{N}(\Phi)$.

Informally, null space vectors must be relatively flat.

Deterministic Sensing Matrices

Showing the NSP for a given Φ involves combinational computational complexity. The coherence of a matrix provides easily computable guarantees.

Coherence

$$\mu(\Phi) = \max_{1 \le i < j \le N} \frac{\left|\langle \Phi_i, \Phi_j \rangle\right|}{\|\Phi_i\| \|\Phi_j\|}$$

Using the coherence it is possible to show that Φ is invertible on the sparse set if:

$$k < \frac{1}{2}\left(1 + \frac{1}{\mu(\Phi)}\right)$$

However, this only guarantees that $k \sim \mathcal{O}(\sqrt{m})$.

Restricted Isometry Property

Low Distortion Embeddings

A useful tool in compressed sensing is the *restricted isometry constant (RIC),* the smallest constant δ_k for which:

$$(1-\delta_k)\|x\|_2 \le \|\Phi x\|_2 \le (1+\delta_k)\|x\|_2$$

holds for all k-sparse vectors x.

A matrix Φ with $\delta_{2k} < 1$ provides an embedding (one-to-one mapping) for the k-sparse set. δ_{2k} also quantifies the robustness of the embedding (low distortion).

Random observations – a key insight in compressed sensing is that random matrices have small RICs with high probability whenever:

$$m \sim \mathcal{O}\left(k\delta_{2k}^{-2}\log_2(N/k)\right)$$

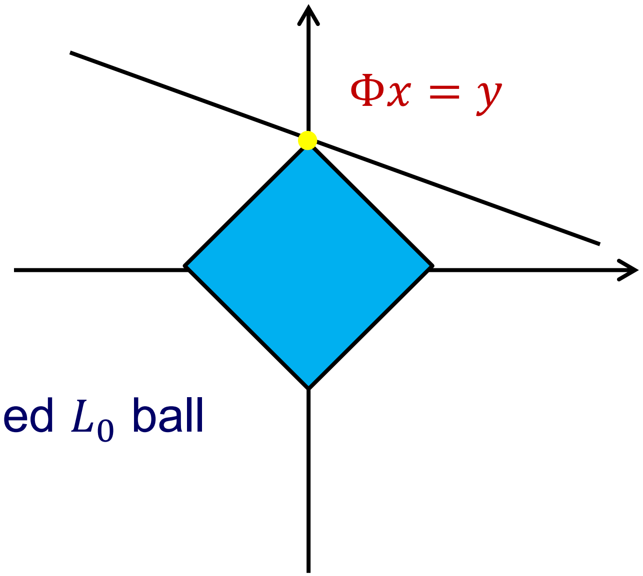
Practical sparse reconstruction

Sparse Recovery via L_1 Minimization

A key advance in Sparse Representations was the use of the L_1 minimization as a proxy for L_0 reconstruction:

$$\hat{x} = \underset{x}{\operatorname{argmin}} \ \|x\|_1 \text{ subject to } \Phi x = y$$

where the L_1 norm is defined as: $\|\boldsymbol{x}\|_1 = \sum_i |x_i|$



Intuition:

1. Minimum L_1 solutions - ● - are sparse
2. L_1 ball is the “closest” convex set to the bounded L_0 ball

L_1 Performance Guarantees

For deterministic matrices L_1 minimization guarantees derived from coherence [Donoho & Elad 2003] : $\mathrm{m} \sim \mathcal{O}(k^2)$.

For general matrices [Candes 2008] showed:

Theorem: If Φ has RIP $\delta_{2k} \leq \sqrt{2} - 1 \implies L_1$NSP $\implies$ Instance Optimality:

$$\|\Delta(\Phi x) - x\|_1 \leq C\sigma_k(x)_1$$

Hence i.i.d. random matrices are near optimal: $\mathrm{m} \sim \mathcal{O}(k \log(N/k))$

Since then it has been shown [Donoho & Tanner 2009] that $L_1 - L_0$ equivalence for sparse vectors if: $\mathrm{m} \geq 2k \log(N/k)$.

Other Practical Recovery Algorithms

The other main class of practical (polynomial complexity) recovery algorithm with performance guarantees is "Greedy methods".

Aim to solve mixed continuous/discrete L_0 minimization problem using:

- Least squares minimization and
- Hard decisions on coefficient selection

There are various flavours with different near optimal guarantees:

- Orthogonal Matching Pursuit (OMP) [Tropp & Gilbert 2007]
- Compressive Sampling Matching Pursuit (CoSAMP) [Needell & Tropp 2008]
- Iterative Hard Thresholding (IHT) [Blumesath & Davies 2009/10]

Performance guarantees come directly from RIP type considerations.

Summary & Engineering Challenges

Sparse Representations provide a powerful nonlinear model for real world signals.

Sparse signals can be sampled and faithfully reconstructed using many fewer samples than predicted by traditional sampling theory.

Engineering Challenges in CS

- What is the right signal model?
  Sometimes obvious, sometimes not. When can we exploit additional structure?
- How can/should we sample?
  Physical constraints; SNR issues; can we randomly sample; exploiting structure; how many measurements?
- What are our application goals?
  Reconstruction? Detection? Estimation?

Selected References

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