Fundamentals of Principal Component Analysis (PCA), Independent Component Analysis (ICA), and Independent Vector Analysis (IVA)

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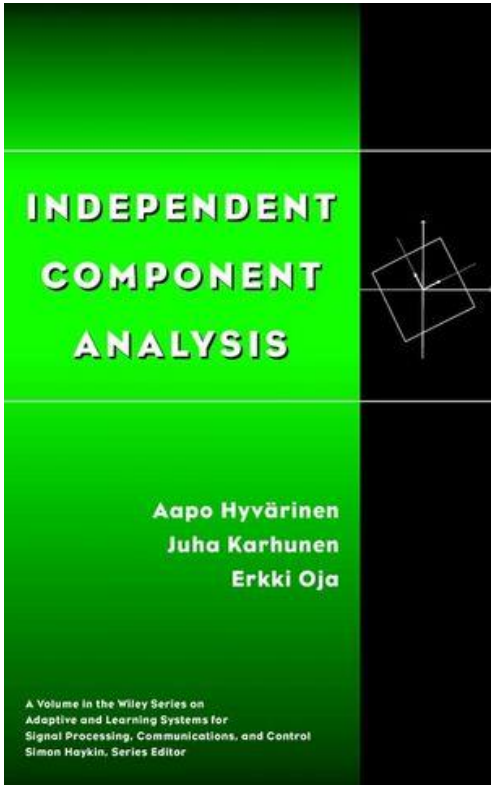
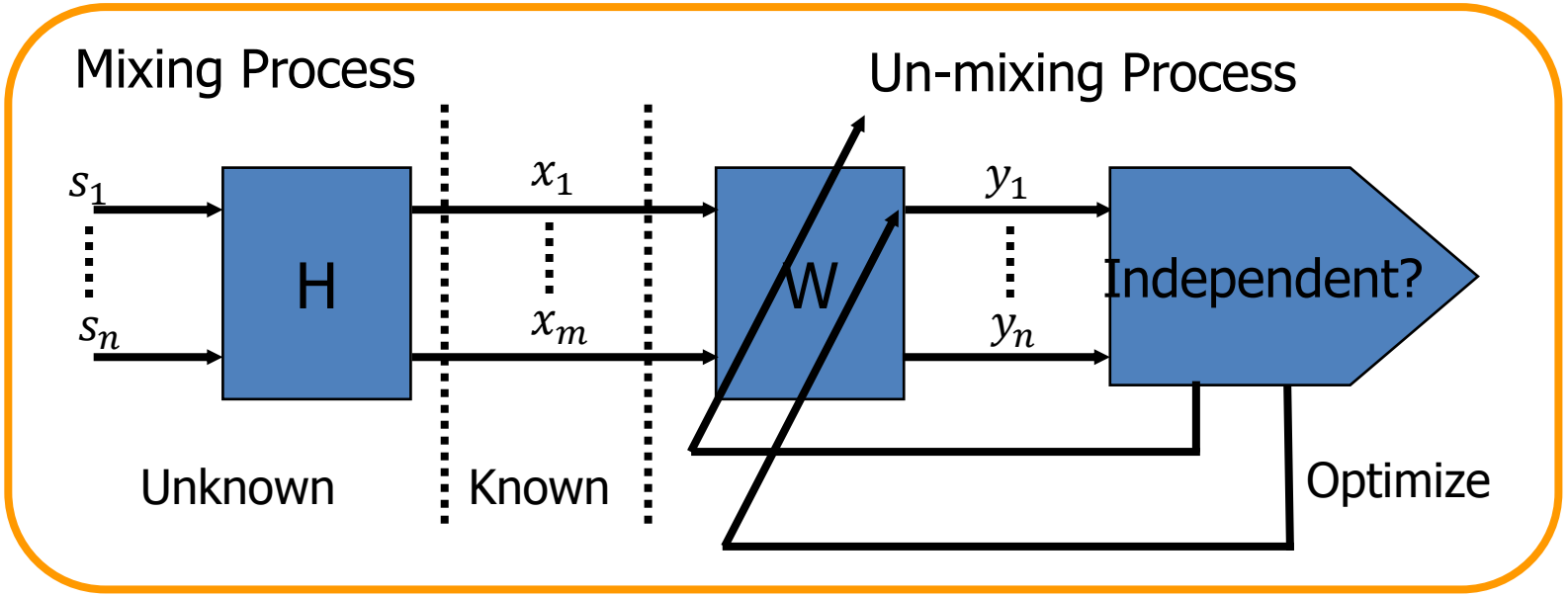
Acknowledgement and Recommended text:

Hyvarinen et al., Independent Component Analysis, Wiley, 2001

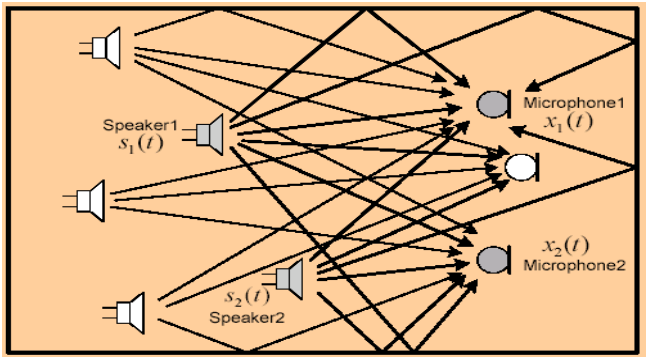


Source Separation and Component Analysis



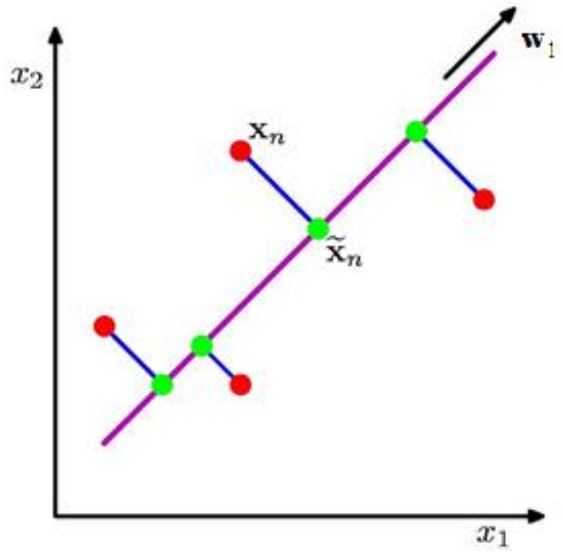
Mixing Model: $\mathrm{x} = \mathrm{Hs}$

Un-mixing Model: $\mathrm{y} = \mathrm{Wx} = \mathrm{WHs} = \mathrm{PDs}$



Principal Component Analysis (PCA)

Orthogonal projection of data onto lower-dimension linear space:

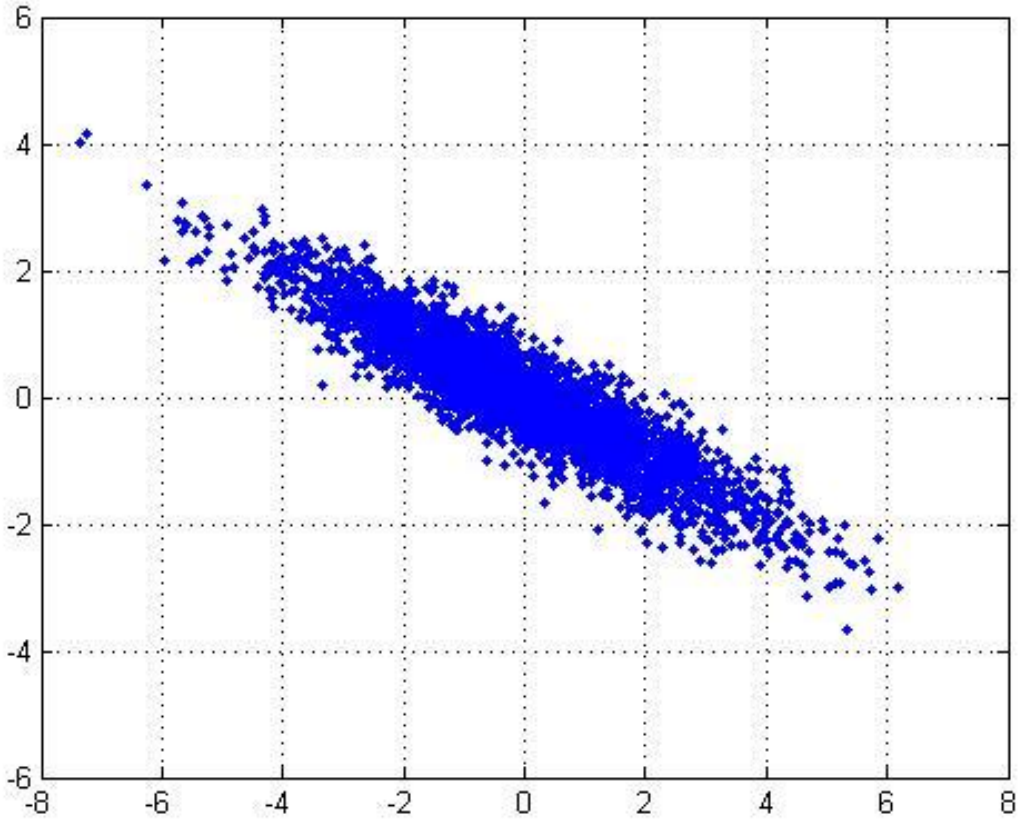


i. maximizes variance of projected data

ii. minimizes mean squared distance between data point and projections

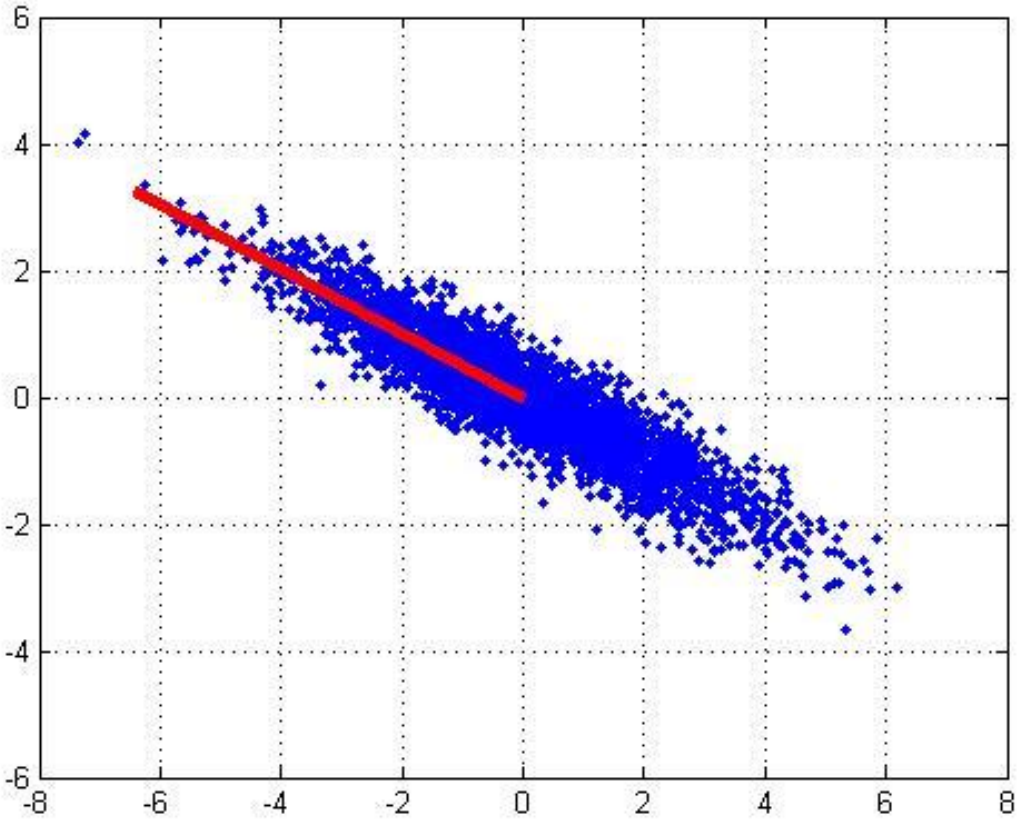
Principal Component Analysis (PCA)

Example: 2-D Gaussian Data



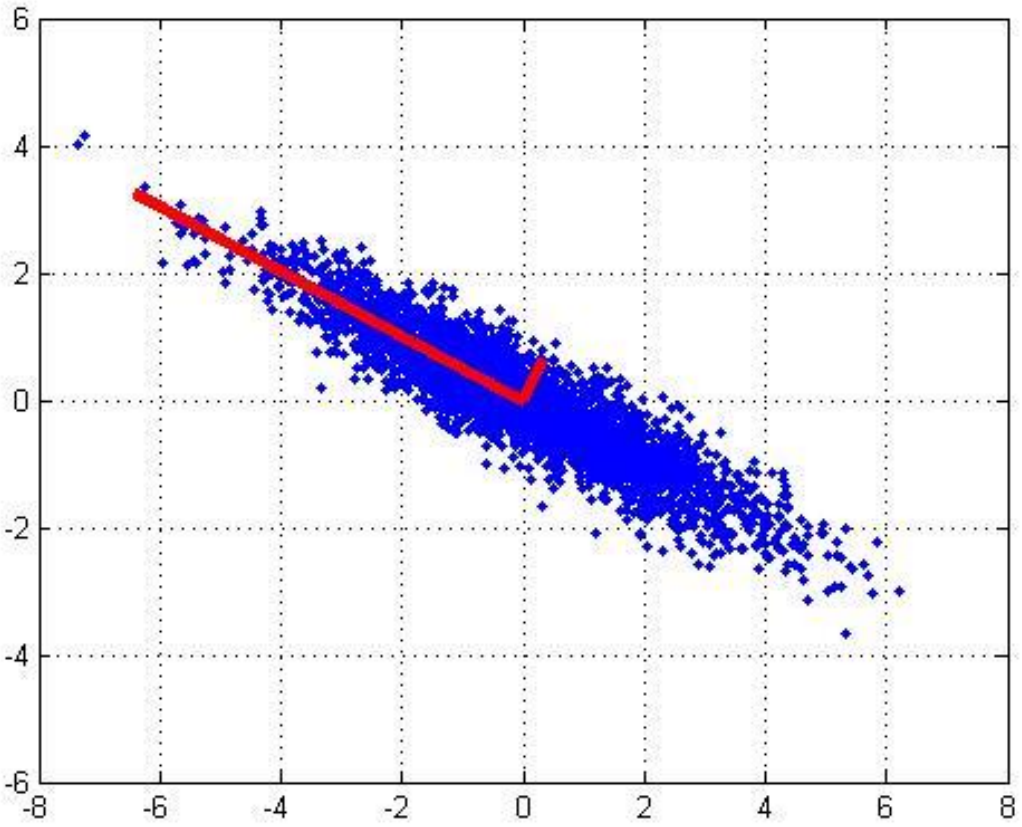
Principal Component Analysis (PCA)

Example: 1st PC in the direction of the largest variance



Principal Component Analysis (PCA)

Example: 2nd PC in the direction of the second largest variance and each PC is orthogonal to other



Principal Component Analysis (PCA)

- In PCA the redundancy is measured by correlation between data elements
- Using only the correlations as in PCA has the advantage that the analysis can be based on second-order statistics (SOS)

In the PCA, first the data is centered by subtracting the mean

$$\mathrm{x} = \mathrm{x} - E\{\mathrm{x}\}$$

Principal Component Analysis (PCA)

The data x is linearly transformed

$$y_1 = \sum_{j=1}^{n} w_{j1} x_j = \mathrm{w}_1^T \mathrm{x}$$

subject to $\|\mathrm{w}_1\| = 1$

Mean of the first transformed factor

$$E\{y_1\} = E\{\mathrm{w}_1^T \mathrm{x}\} = \mathrm{w}_1^T E\{\mathrm{x}\} = 0$$

Variance of the first transformed factor

$$E\{y_1^2\} = E\{(\mathrm{w}_1^T \mathrm{x})^2\} = E\{(\mathrm{w}_1^T \mathrm{x})(\mathrm{x}^T \mathrm{w}_1)\}$$

Principal Component Analysis (PCA)

$$E\{y_1^2\} = E\{(\mathrm{w}_1^T \mathrm{x})(\mathrm{x}^T \mathrm{w}_1)\} = \mathrm{w}_1^T E\{\mathrm{x}\mathrm{x}^T\}\,\mathrm{w}_1$$

$$= \mathrm{w}_1^T \mathrm{R}_{\mathrm{xx}}\,\mathrm{w}_1 \qquad (1)$$

The above correlation matrix is symmetric

$$\mathrm{a}^T \mathrm{R}_{\mathrm{xx}}\,\mathrm{b} = \mathrm{b}^T \mathrm{R}_{\mathrm{xx}}\,\mathrm{a}$$

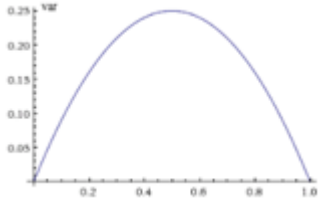
Correlation matrix R_{xx} is not in our control and depends on the data x.

We can control w?

Principal Component Analysis (PCA)

(PCA by Variance Maximization)

At maximum variance a small change in w will not affect the variance



$$E\{y_1^2\} = (\mathrm{w}_1 + \Delta\mathrm{w})^T \mathrm{R}_{\mathrm{XX}} (\mathrm{w}_1 + \Delta\mathrm{w})$$

$$= \mathrm{w}_1^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_1 + 2\Delta\mathrm{w}^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_1 + \Delta\mathrm{w}^T \mathrm{R}_{\mathrm{XX}} \Delta\mathrm{w}$$

where Δw is very small quantity and therefore

$$E\{y_1^2\} = \mathrm{w}_1^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_1 + 2\Delta\mathrm{w}^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_1 \qquad (2)$$

By using (1) and (2) we can write

$$\Delta\mathrm{w}^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_1 = 0 \qquad (3)$$

Principal Component Analysis (PCA)

As we know that

$$\| w_1 + \Delta w \| = 1$$

therefore

$$(w_1 + \Delta w)^T (w_1 + \Delta w) = 1$$

$$w_1^T w_1 + 2\Delta w^T w_1 + \Delta w^T \Delta w = 1$$

where Δw is very small quantity and therefore

$$1 + 2\Delta w^T w_1 + 0 = 1$$

$$\Delta w^T w_1 = 0 \qquad (4)$$

The above result shows that w_1 and Δw are orthogonal to each other.

Principal Component Analysis (PCA)

By careful comparison of (3) and (4) and by considering $\mathrm{R_{XX}}$

$$\Delta \mathrm{w}^T \mathrm{R_{XX}}\, \mathrm{w}_1 - \Delta \mathrm{w}^T \lambda_1 \mathrm{w}_1 = 0$$

$$\Delta \mathrm{w}^T [\mathrm{R_{XX}}\, \mathrm{w}_1 - \lambda_1 \mathrm{w}_1] = 0$$

since $\Delta \mathrm{w}^T \neq 0$

$$\therefore \quad [\mathrm{R_{XX}}\, \mathrm{w}_1 - \lambda_1 \mathrm{w}_1] = 0$$

$$\rightarrow \quad \mathrm{R_{XX}}\, \mathrm{w}_i = \lambda_i \mathrm{w}_i \qquad i = 1, 2, \ldots, n \qquad (5)$$

where

$\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots, \lambda_n$ are eigenvalues of $\mathrm{R_{XX}}$

and

$\mathrm{w}_1, \mathrm{w}_2, \ldots\ldots\ldots\ldots, \mathrm{w}_n$ are eigenvectors of $\mathrm{R_{XX}}$

Principal Component Analysis (PCA)

$$R_{XX}\, w_i = \lambda_i w_i , \quad i = 1, 2, \ldots, n$$

$$\lambda_1 = \lambda_{max} > \lambda_2 > \ldots\ldots\ldots > \lambda_n$$

and

$$E = [\, w_1, w_2, \ldots\ldots\ldots, w_n]$$

$$\therefore \quad R_{XX} E = E\, \Lambda \qquad (6)$$

where

$$\Lambda = \text{dig}\, [\lambda_1,\ \lambda_2, \ldots\ldots\ldots, \lambda_n]$$

We know that E is orthogonal matrix

$$E^T E = I$$

and we can update (7)

$$E^T R_{XX} E = E^T E\, \Lambda = I\Lambda = \Lambda \qquad (7)$$

Principal Component Analysis (PCA)

In expanded form (7) is

$$\mathrm{w}_i^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_j = \begin{bmatrix} \lambda_i & i = j \\ 0 & i \neq j \end{bmatrix}$$

R_{XX} is symmetric and positive definite matrix and can be represented as [Hyvarinen et al.]

$$\mathrm{R}_{\mathrm{XX}} = \sum_{i=1}^{n} \lambda_i \, \mathrm{w}_i^T \, \mathrm{w}_j$$

We know

$$\mathrm{w}_i^T \mathrm{w}_j = \begin{bmatrix} 1 & i = j \\ 0 & i \neq j \end{bmatrix}$$

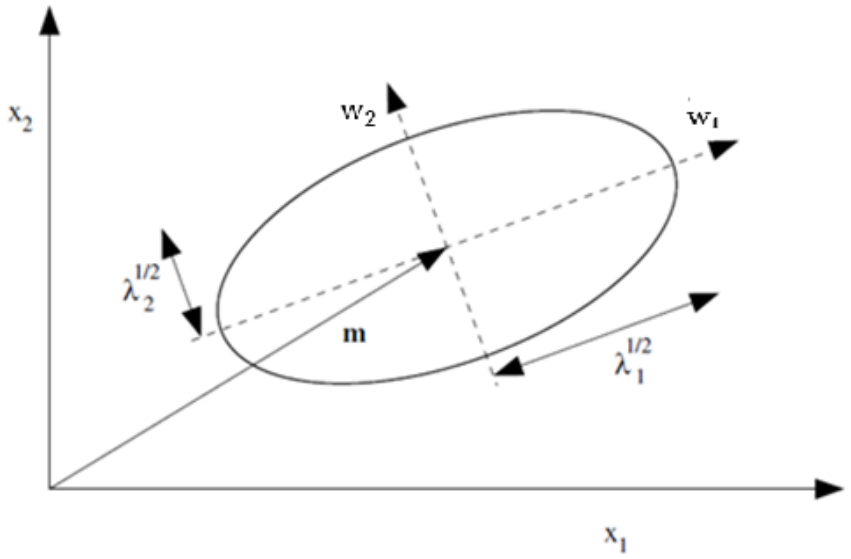
Finally, (1) will become

$$E\{y_i^2\} = \mathrm{w}_i^T \mathrm{R}_{\mathrm{XX}} \mathrm{w}_i = \lambda_i, \qquad i = 1, 2, \ldots, n \qquad (8)$$

Can we define the principal components now?

Principal Component Analysis (PCA)

In a multivariate Gaussian probability density, the principal components are in the directions of eigenvectors w_i and the respective variances are the eigenvalues λ_i.



There are n possible solutions for w and

$$y_i = \mathrm{w}_i^T \mathrm{x} = \mathrm{x}^T \mathrm{w}_i \qquad i = 1, 2, \ldots, n$$

Principal Component Analysis (PCA)

Whitening

Data whitening is another form of PCA

The objective of whitening is to transform the observed vector $z = Vx$, which is uncorrelated and with variance equal to identity matrix

$$E\{zz^T\} = I$$

The unmixing matrix, W, can be decomposed into two components:

$$W = UV$$

Where U is rotation matrix and V is the whitening matrix and

$$V = \Lambda^{-1/2} E^T$$

Principal Component Analysis (PCA)

Whitening

$$\mathrm{z} = \mathrm{Vx}$$

$$E\{\mathrm{zz}^T\} = \mathrm{V}E\{\mathrm{xx}^T\}\mathrm{V}^T = \Lambda^{-1/2}\,\mathrm{E}^T E\{\mathrm{xx}^T\}\mathrm{E}\,\Lambda^{-1/2}$$

$$= \Lambda^{-1/2}\,\mathrm{E}^T \mathrm{R}_{\mathrm{xx}}\mathrm{E}\,\Lambda^{-1/2} = \mathrm{I}$$

The covariance is unit matrix, therefore z is whitened

Whinintng matrix, V, is not unique. It can be pre-multiply by an orthogonal matrix to obtain another version of V

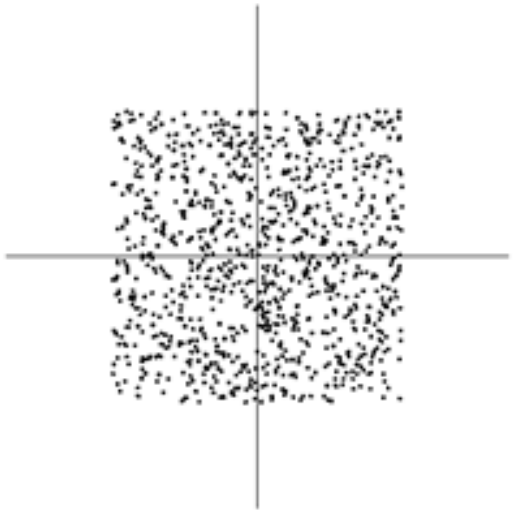
Limitations:

PCA only deals with second order statistics and provides only decorrelation. Uncorrelated components are not independent

Principal Component Analysis (PCA)

Limitations

Two components with uniform distributions and their mixture



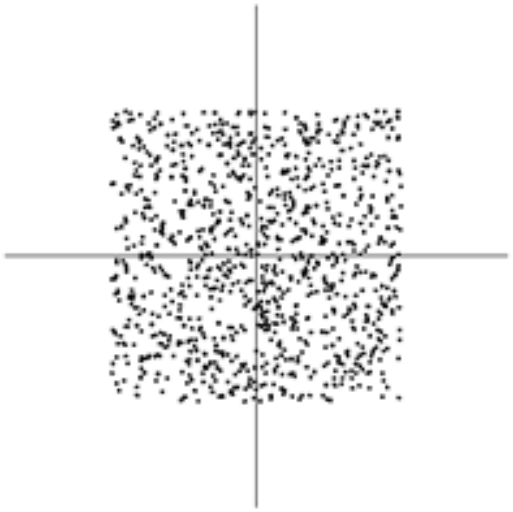
The joint distribution of ICs s_1(horizontal axis) and s_2 (vertical axis) with uniform distributions.

The joint distribution of observed mixtures x_1(horizontal axis) and x_2 (vertical axis).

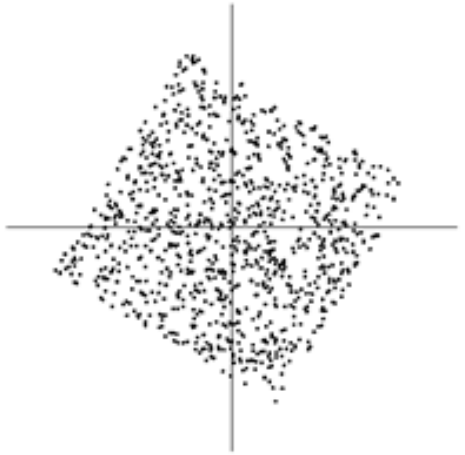
Principal Component Analysis (PCA)

Limitations

Two components with uniform distributions and after PCA



The joint distribution of ICs s_1(horizontal axis) and s_2 (vertical axis) with uniform distributions.



The joint distribution of the whitened mixtures of uniformly distributed ICs

PCA does not find original coordinates

- Factor rotation problem

Principal Component Analysis (PCA)

Limitations: uncorrelation and independence

If two zero mean random variables s_1 and s_2 are uncorrelated then their covariance is zero:

$$\text{cov}(s_1, s_2) = E\{s_1 s_2\} = 0$$

If s_1 and s_2 are independent, then for any two functions, $g_1$and g_2 we have

$$E\{g_1(s_1)g_2(s_2)\} = E\{g_1(s_1)\}E\{g_2(s_2)\}$$

If random variables are independent then they are also uncorrelated

If s_1 and s_2 are discrete valued and follow such a distribution that the pair are with probability 1/4 equal to any of the following values: (0,1), (0,-1), (1,0) and (-1,0). Then

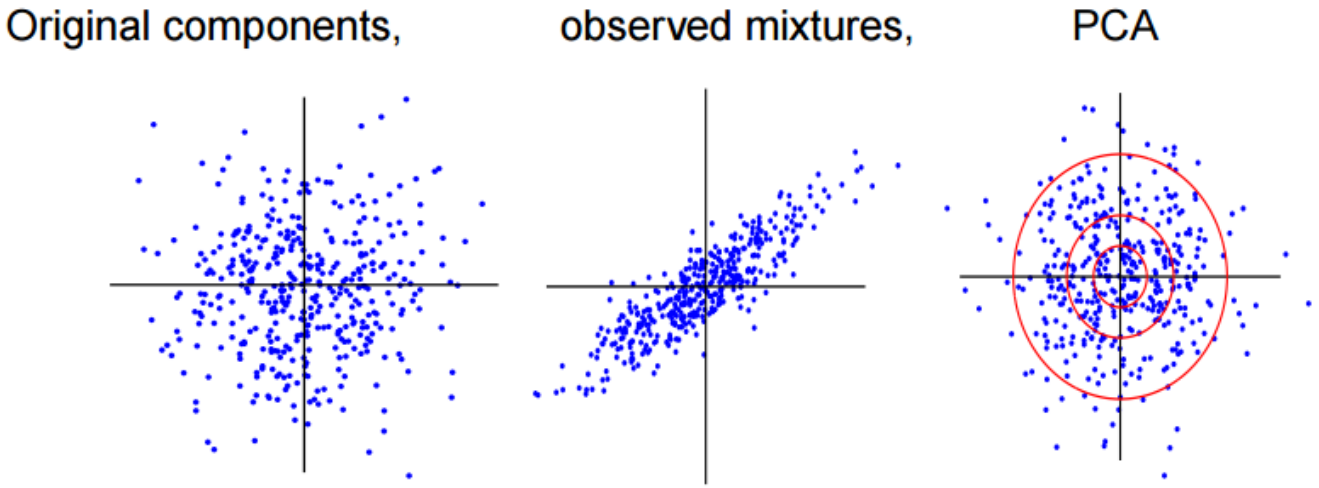
$$E\{\, s_1^2 s_2^2)\} = 0 \;\neq \frac{1}{4} = E\{s_1^2\}E\{s_2^2\}$$

Therefore, uncorrelated components are not independent

Principal Component Analysis (PCA)

Limitations: uncorrelation and independence

However, uncorrelated components of Gaussian distribution are independent

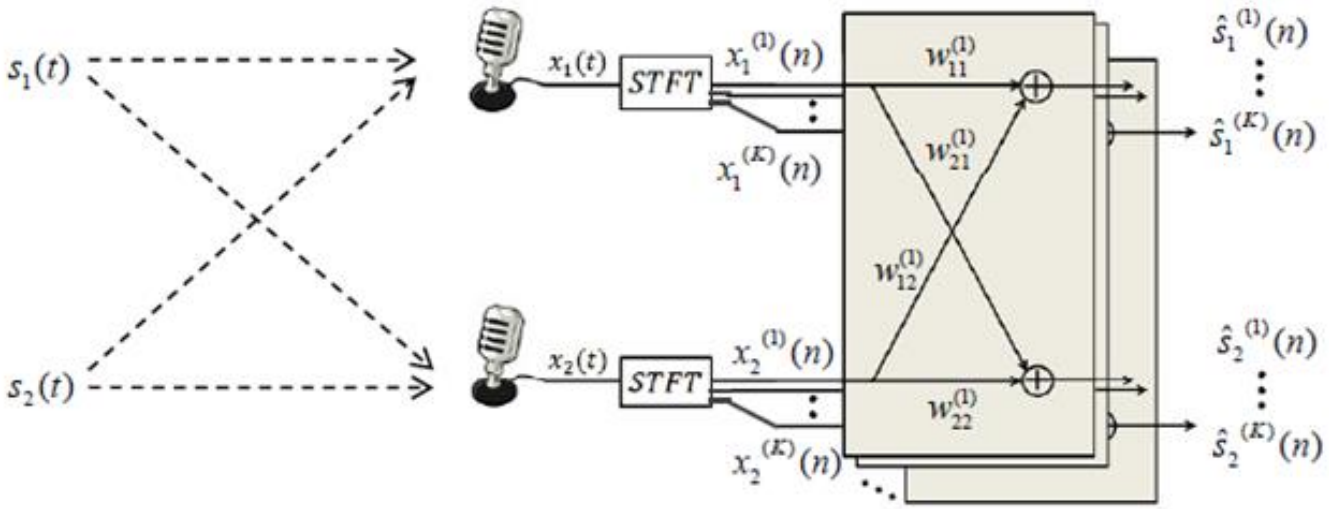


Distribution after PCA is same as distribution before mixing

Independent Component Analysis (ICA)

Independent component analysis (ICA) is a statistical and computational technique for revealing hidden factors that underlie sets of random variables, measurements, or signals.

ICA separates the sources at each frequency bin



Independent Component Analysis (ICA)

The fundamental restrictions in ICA are:

i. The sources are assumed to be statistically independent of each other. Mathematically, independence implies that the joint probability density function p(s) of the sources can be factorized

$$p(s) = \prod_{i=1}^{n} p_i(s_i)$$

where $p_i(s_i)$ is the marginal distribution of the i-th source

ii. All but one of the sources must have non-Gaussian distributions

Independent Component Analysis (ICA)

Why?

The joint pdf of two Gaussian ICs, s_1 and s_2 is:

$$p(s_1, s_2) = \frac{1}{2\pi}\exp\left(-\frac{s_1^2+s_2^2}{2}\right) = \frac{1}{2\pi}\exp\left(-\frac{\|\mathbf{s}\|^2}{2}\right)$$

If the mixing matrix H is orthogonal, the joint pfd of the mixtures x_1 and x_2 is:

$$p(x_1, x_2) = \frac{1}{2\pi}\exp\left(-\frac{\|\mathbf{H}^T\mathbf{s}\|^2}{2}\right)\left|\det\mathbf{H}^T\right|$$

H is orthogonal, therefore $\left\|\mathbf{H}^T\mathbf{s}\right\|^2 = \|\mathbf{s}\|^2$ and $\left|\det\mathbf{H}^T\right| = 1$

$$p(x_1, x_2) = \frac{1}{2\pi}\exp\left(-\frac{\|\mathbf{s}\|^2}{2}\right)$$

Orthogonal mixing matrix does not change the pdf and original and mixing distributions are identical. Therefore, there is no way to infer the mixing matrix from the mixtures

Independent Component Analysis (ICA)

iii. In general, the mixing matrix H is square ($n = m$) and invertible

iv. Methods to realize ICA are more sensitive to data length than the methods based on SOS

Mainly, ICA relies on two steps:

A statistical criterion, e.g. nongaussianity measure, expressed in terms of a cost/contrast function $J(g(y))$

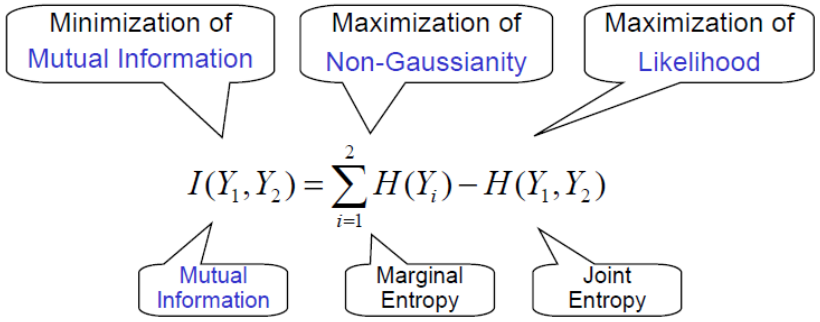
An optimization technique to carry out the minimization or maximization of the cost function $J(g(y))$

Independent Component Analysis (ICA)

1. Minimization of Mutual Information
   (Minimization of Kullback-Leibler Divergence)

2. Maximization of Likelihood

3. Maximization of Nongaussianity



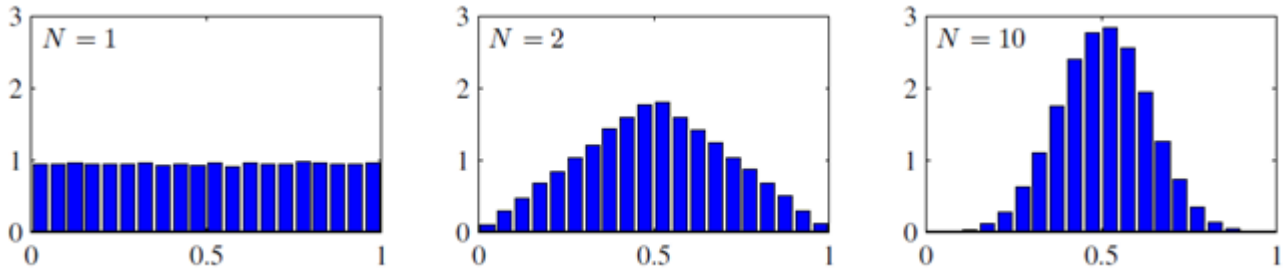
$H(\cdot)$: Entropy

- All solutions are identical

Independent Component Analysis (ICA)

Nongaussainity and Independence

Central limit theorem: subject to certain mild conditions, the sum of a set of random variables has a distribution that becomes increasingly Gaussian as the number of terms in the sum increases



Histogram plots of the mean of N uniformly distributed numbers for various values of N. As N increases, the joint distribution tends towards a Gaussian.

So roughly, any mixture of components will be more Gaussian than the components themselves

Independent Component Analysis (ICA)

Nongaussainity and Independence

Central limit theorem: The distribution of Independent random variables tends towards a Gaussian distribution, under certain conditions

If z_i is independent and identically distributed random variable then $x_n = \sum_{i=1}^{n} z_i$

Mean and variance of x_n grow without bound when $n \to \infty$, and $y_n = (x_n - \mu_{x_n})/\delta_{x_n}^2$

It has been shown that the distribution of y_n converges to a Gaussian distribution with zero mean and unit variance when $n \to \infty$. This is known as the central limit theorem

Independent Component Analysis (ICA)

Nongaussainity and Independence

In BSS, if $s_j, j = 1, \ldots, n$ are unknown source signals and mixed with coefficients $h_{ij}, i = 1, \ldots, m$ then

$$x_i = \sum_{j=1}^{n} h_{ij} s_j$$

The distribution of the mixture x_i is usually near to Gaussian when the number of sources s_j is fairly small

We know that

$$y_i = \sum_{j=1}^{m} w_{ji} x_j = \mathrm{w}_i^T \mathrm{x}$$

How could the central limit theorem be used to determine w_i so that it would equal to one of the rows of the inverse of mixing matrix H ($\mathrm{y} \approx \mathrm{H}^{-1}\mathrm{x}$) ?

Independent Component Analysis (ICA)

Nongaussainity and Independence

In practice, we couldn't determine such a w_i exactly, because the problem is blind i.e. H is unknown

We can find an estimator that gives good approximation of w_i.

By varying the coefficients in w_i, and monitoring the change in distribution of $y_i = w_i^T x$

Hence, maximizing the non- Gaussainity of $w_i^T x$ provides one of the Independent Components

How we can maximize the nongaussainity ?

Independent Component Analysis (ICA)

Nongaussainity by Kurtosis

For ICA estimation based on nongaussainity, we require a qualitative measures of nongaussainity of a random variable

Kurtosis is the name given to the forth-order cumulants of a random variable e.g.y, the kurtosis is defined as:

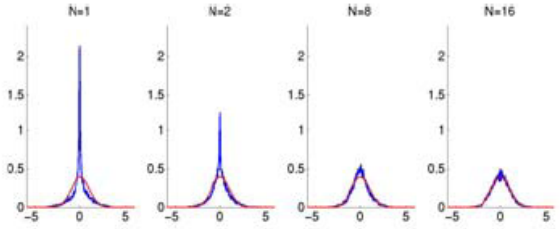
$$\text{kurt}(y) = E\{y^4\} - 3(E\{y^2\})^2$$

If the data is whitened, e.g. zero mean and unit variance, then $\text{kurt}(y) = E\{y^4\} - 3$ is normalized version of the fourth moment $E\{y^4\}$

Independent Component Analysis (ICA)

Nongaussainity by Kurtosis

For Gaussian variable y, the fourth moment is equal to $3(E\{y^2\})^2$. Therefore, kurtosis is zero for Gaussian random variables



N	1	2	8	16		Gaussian
Kurtosis	2.1	1.8	0.70	0.39		0

$$kurt(y) = E\{|y|^4\} - 3(E\{|y|^2\})^2$$

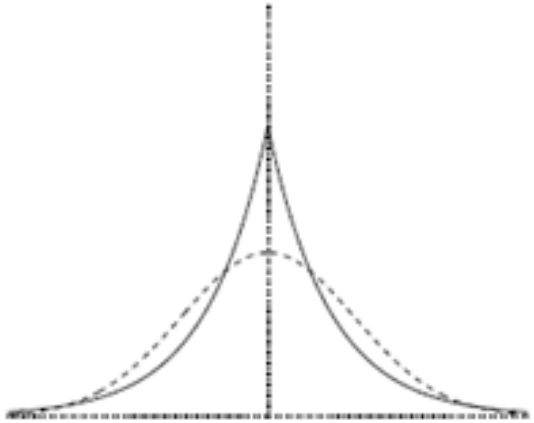
Typically nongaussianity is measured by the absolute value of kurtosis

Independent Component Analysis (ICA)

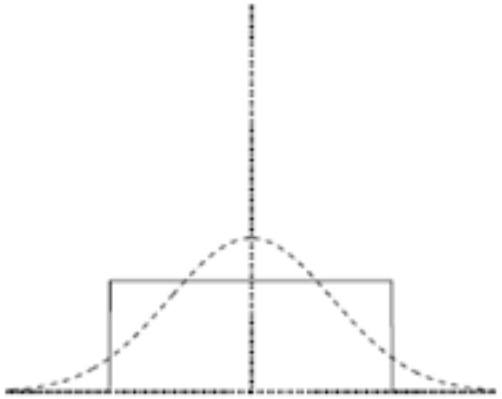
Nongaussian Distributions

Random variable that have a positive kurtosis are called supergaussian.
A typical example is Laplacian distribution

Random variable that have a negative kurtosis are called subgaussian.
A typical example is the uniform distribution



Laplacian and Gaussian distributions.



Uniform and Gaussian distributions.

Independent Component Analysis (ICA)

Limitations of Kurtosis

Kurtosis is not a robust measure of nongaussainity

Kurtosis can be very sensitive to outliers. For example, in 1000 samples of a random variable of zero mean and unit variance, if one value is equal to 10, then

$$\text{kurt}(y) = \frac{10^4}{1000} - 3 = 7$$

A single value can make kurtosis large and the value of kurtosis may depends only on few observations in the tail of the distribution

We have a better measure of nongaussainity than kurtosis?

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

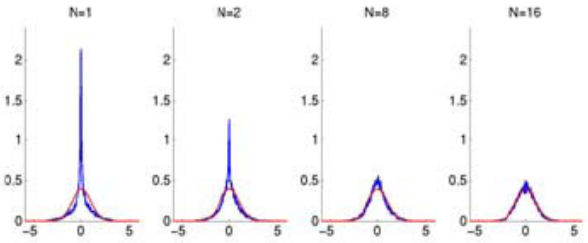
A fundamental result of information theory is that a Gaussian variable has the largest entropy among all variables of equal variance.

A robust but computationally complicated measure of nongaussainity is negentropy

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

A measure that is zero for Gaussian variables and always nonnegative can be obtained from differential entropy, and called negentropy



# sources N	1	2	8	16	Gaussian
Entropy *H*	1.19	1.33	1.39	1.40	1.41
Negentropy *N*	0.225	0.087	0.025	0.012	0

$$H(y) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$$

$$N(y) = H(x_{\text{gauss}}) - H(y)$$

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

A classical method of approximating negentropy based on higher-order cumulants for zero mean random variable y is:

$$J(y) \approx \frac{1}{12} E\{y^3\}^2 + \frac{1}{48} \text{kurt}(y)^2$$

For symmetric distributions, the first term on right hand side of the above equation is zero.

Therefore, a more sophisticated approximation of negentropy is required

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

The method that uses nonquadratic function G, and provides a simple way of approximating the negentropy is [Hyvarinen et al]:

$$J(y) \propto [E\{G(y)\} - E\{G(v)\}]^2$$

where v is a standard random Gaussian variable

By choosing G wisely, we can obtain approximation of negentropy that is better than the one provided by kurtosis

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

By choosing a G that does not grow too fast, we can obtain more robust estimator.

The following choices of G have proved that these are very useful

$$G_1(y) = \frac{1}{a_1}\log\cosh a_1 y, \qquad G_2(s) = \exp(-\frac{y^2}{2})$$

where $1 < a_1 < 2$ is a constant and mostly equal to one

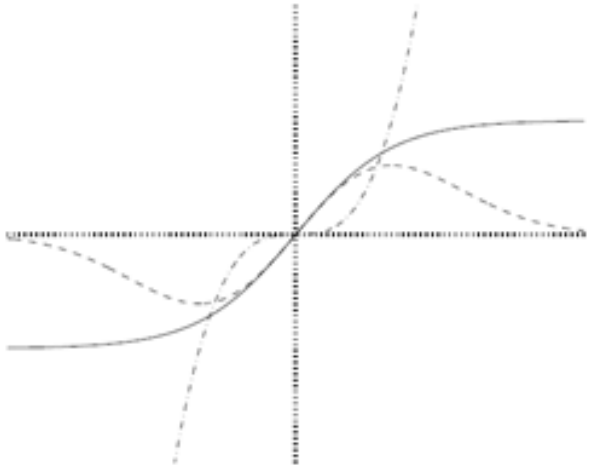
The derivatives of above contrast functions are used in ICA algorithms

$$g_1(y) = \tanh(a_1 y),$$
$$g_2(y) = y\exp(-\frac{y^2}{2}),$$
$$g_3(y) = y^3$$

Independent Component Analysis (ICA)

Nongaussainity by Negentropy

Different nonlinearities are required, which depends on the distribution of ICs



The robust nonlinearities g_1 (solid line), g_2 (dashed line) and g_3 (dash-doted line).

Can we optimize the local maxima for nongaussainity of a linear combination $y = \sum_i w_i x_i$ under the constraint that the variance of y is constant?

Independent Component Analysis (ICA)

ICA Estimation Principle 1

To find the unmixing matrix W, so that the components y_i and y_j, where $i \neq j$, are uncorrelated, and the their nonlinear functions $g(y_i)$ and $h(y_j)$, are also uncorrelated.

The above nonlinear decorrelation is the basic ICA method

Independent Component Analysis (ICA)

ICA Learning Rules

In gradient descent, we minimize a cost function $J(\mathrm{W})$ from its initial point by computing its gradient at that point, and then move in the direction of steepest descent

The update rule with the gradient taken at the point W_k is:

$$\mathrm{W}_{k+1} = \mathrm{W}_k - \mu \frac{\partial J(\mathrm{W})}{\partial \mathrm{W}}$$

According to Amari et al., the largest increase in $J(\mathrm{W}+\partial \mathrm{W})$ can be obtained in the direction of natural gradient

$$\frac{\partial J(\mathrm{W})}{\partial \mathrm{W}_{nat}} = \frac{\partial J(\mathrm{W})}{\partial \mathrm{W}} \mathrm{W}^T \mathrm{W}$$

Independent Component Analysis (ICA)

ICA Learning Rules

Therefore, the update equation of natural gradient descent is:

$$\mathrm{W}_{k+1} = \mathrm{W}_k - \mu \frac{\partial J(\mathrm{W})}{\partial \mathrm{W}} \mathrm{W}_k^T \mathrm{W}_k$$

The final update equation of the natural gradient algorithm is:

$$\mathrm{W}_{k+1} = \mathrm{W}_k + \mu[\mathrm{I} - \mathrm{y}^T g(\mathrm{y})]\, \mathrm{W}_k$$

where μ is called the step size or learning rate

For stability and convergence of the algorithm $0 < \mu \ll \infty$. If we select a very small value of μ then it will take more time to reach local minima or maxima

Independent Component Analysis (ICA)

ICA Learning Rule

Initialize the unmixing matrix

$$\mathrm{W}_0 = \mathrm{randn}(n, n)$$

For $n = 2$

$$\mathrm{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad g(\mathrm{Y}) = \begin{bmatrix} g(y_1) \\ g(y_2) \end{bmatrix}$$

The target is:

$$[\mathrm{I} - \mathrm{y} g(\mathrm{y})^T] = 0$$

$$\Rightarrow E\{\mathrm{I}\} = E\{\mathrm{y} g(\mathrm{y})^T\} = \mathrm{R}_{yy}$$

How we can diagonalize R_{yy}?

Independent Component Analysis (ICA)

ICA Learning Rule

If

$$\Delta W = W_{k+1} - W_k$$

and

$$R_{yy} = \begin{bmatrix} y_1 g(y_1) & y_1 g(y_2) \\ y_2 g(y_1) & y_1 g(y_2) \end{bmatrix},$$

Then

$$\Delta W = \mu \begin{bmatrix} c_1 - y_1 g(y_1) & y_1 g(y_2) \\ y_2 g(y_1) & c_2 - y_1 g(y_2) \end{bmatrix} W \Longrightarrow 0$$

Update W so that y_1 and y_2 become mutually independent

Independent Component Analysis (ICA)

ICA Estimation Principle 2

To find the local maxima of nongaussainity of a linear combination $y = \sum_i w_i x_i$ under the condition that the variance of y is constant. Each local maxima will provide one independent component

ICA by maximization of nongaussainity is one of the most widely used techniques

Independent Component Analysis (ICA)

ICA Learning Rule

We can derive a simple gradient algorithm to maximize negentropy

First we whiten the data $\mathrm{z} = \mathrm{Vx}$, where V is the whitening matrix

We already know that the function to measure negentropy is:

$$J(y) \propto [E\{G(y)\} - E\{G(v)\}]^2$$

We know that $y = \mathrm{w}^T\mathrm{z}$ and take the derivative of $J(y)$ with respect to w

$$\frac{\partial J(y)}{\partial \mathrm{w}} \propto$$
$$2[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]\frac{\partial}{\partial w}[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]$$

Independent Component Analysis (ICA)

ICA Learning Rule

$$\frac{\partial J(y)}{\partial \mathrm{w}} \propto 2[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]\frac{\partial}{\partial w}[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]$$

$$\propto 2[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]E\{g(\mathrm{w}^T\mathrm{z})\frac{\partial}{\partial w}(\mathrm{w}^T\mathrm{z})\}$$

$$\propto 2[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\}$$

If $\gamma = 2[E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}]$ then

$$\frac{\partial J(y)}{\partial \mathrm{w}} \propto \gamma\, E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\}$$

Hence, the update equations of a gradient algorithm are:

$$\Delta\mathrm{w} \propto \gamma\, E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} \quad \text{and} \quad \mathrm{w} \leftarrow {}^{\mathrm{w}}/_{\|\mathrm{w}\|}$$

Independent Component Analysis (ICA)

Implementation Summary

i. Center the data $\mathrm{x} = \mathrm{x} - E\{\mathrm{x}\}$

ii. Whiten the data $\mathrm{z} = \mathrm{V}\mathrm{x}$

iii. Select a nonlinearity g

iv. Randomly initialize w, with $\|\mathrm{w}\| = 1$, and γ

v. Update $\Delta\mathrm{w} \propto \gamma\, E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\}$

vi. Normalize $\mathrm{w} \leftarrow {}^{\mathrm{w}}/_{\|\mathrm{w}\|}$

vii. Update, if required, $\Delta\gamma \propto [E\{G(\mathrm{w}^T\mathrm{z})\} - E\{G(v)\}] - \gamma$

Repeat from step-v, if not converged.

Independent Component Analysis (ICA)

Fast Fixed-point Algorithm (FastICA)

FastICA is based on a fixed point iteration scheme to find a maximum of the nongaussianity of IC y

The above gradient method suggests the following fixed point iteration

$$\mathrm{w} = E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\}$$

the coefficient γ is eliminated by the normalization step

We can modify the iteration by multiplying w with some constant α and add on both sides of the above equation

$$(1+\alpha)\mathrm{w} = E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} + \alpha\,\mathrm{w}$$

Independent Component Analysis (ICA)

Fast Fixed-point Algorithm (FastICA)

$$(1+\alpha)\mathrm{w} = E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} + \alpha\ \mathrm{w}$$

Due to the subsequent normalization of w to unit norm, the above equation gives a fixed point iteration. And α plays an important role in fast convergence

The optima of $E\{G(\mathrm{w}^T\mathrm{z})\}$ under the constraint $E\{G(\mathrm{w}^T\mathrm{z})^2\} = \|\mathrm{w}\|^2 = 1$ are obtained at points when the gradient of the Lagrangian is zero [Hyvarinen et al.]

$$\frac{\partial J(\mathrm{y})}{\partial \mathrm{w}} = E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} + \beta\ \mathrm{w}$$

We can take derivative of the above equation to find the local maxima

$$\frac{\partial^2 J(\mathrm{y})}{\partial \mathrm{w}^2} = E\{\mathrm{z}\mathrm{z}^T g'(\mathrm{w}^T\mathrm{z})\} + \beta\ \mathrm{I}$$

Independent Component Analysis (ICA)

Fast Fixed-point Algorithm (FastICA)

Data is whitened therefore

$$\frac{\partial^2 J(\mathrm{y})}{\partial \mathrm{w}^2} = E\{g'(\mathrm{w}^T \mathrm{z})\}\mathrm{I} + \beta\ \mathrm{I}$$

According to Newton's method

$$\mathrm{w} \leftarrow \mathrm{w} - \left[\frac{\partial^2 J(\mathrm{y})}{\partial \mathrm{w}^2}\right]^{-1} \frac{\partial J(\mathrm{y})}{\partial \mathrm{w}}$$

Therefore Newton iteration is:

$$\mathrm{w} \leftarrow \mathrm{w}\ - [E\{g'(\mathrm{w}^T \mathrm{z})\} + \beta]^{-1}\ (E\{\mathrm{z}g(\mathrm{w}^T \mathrm{z})\} + \beta\ \mathrm{w})$$

Independent Component Analysis (ICA)

Fast Fixed-point Algorithm (FastICA)

$$\mathrm{w} \leftarrow \mathrm{w} - [E\{g'(\mathrm{w}^T\mathrm{z})\} + \beta]^{-1} (E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} + \beta\, \mathrm{w})$$

Multiply both sides of above equation with $E\{g'(\mathrm{w}^T\mathrm{z})\} + \beta$ and by simplifying, we obtain basic fixed-point iteration in FastICA

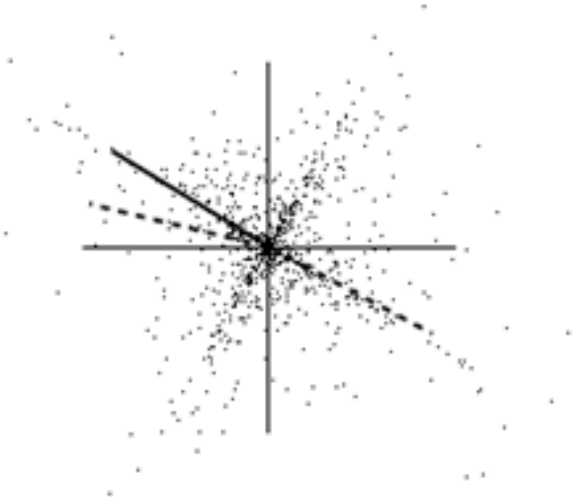
$$\mathrm{w} \leftarrow E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} - E\{g'(\mathrm{w}^T\mathrm{z})\}$$

and

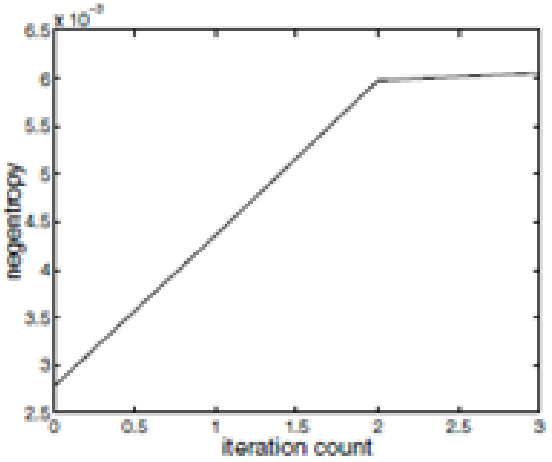
$$\mathrm{w} \leftarrow {}^{\mathrm{w}}/_{\|\mathrm{w}\|}$$

Independent Component Analysis (ICA)

Fast Fixed-point Algorithm (FastICA)



Results with FastICA using negentropy. Doted and solid lines show w after first and second iterations respectively (not at actual scale).



The convergence of FastICA using negentropy, for supergusasain ICs (not at actual scale)

Independent Component Analysis (ICA)

Implementation Summary of FastICA Algorithm

i. Center the data $\mathrm{x} = \mathrm{x} - E\{\mathrm{x}\}$

ii. Whiten the data $\mathrm{z} = \mathrm{V}\mathrm{x}$

iii. Select a nonlinearity g

iv. Randomly initialize w, with constraint $\|\mathrm{w}\| = 1$

v. Update $\mathrm{w} \leftarrow E\{\mathrm{z}g(\mathrm{w}^T\mathrm{z})\} - E\{g'(\mathrm{w}^T\mathrm{z})\}$

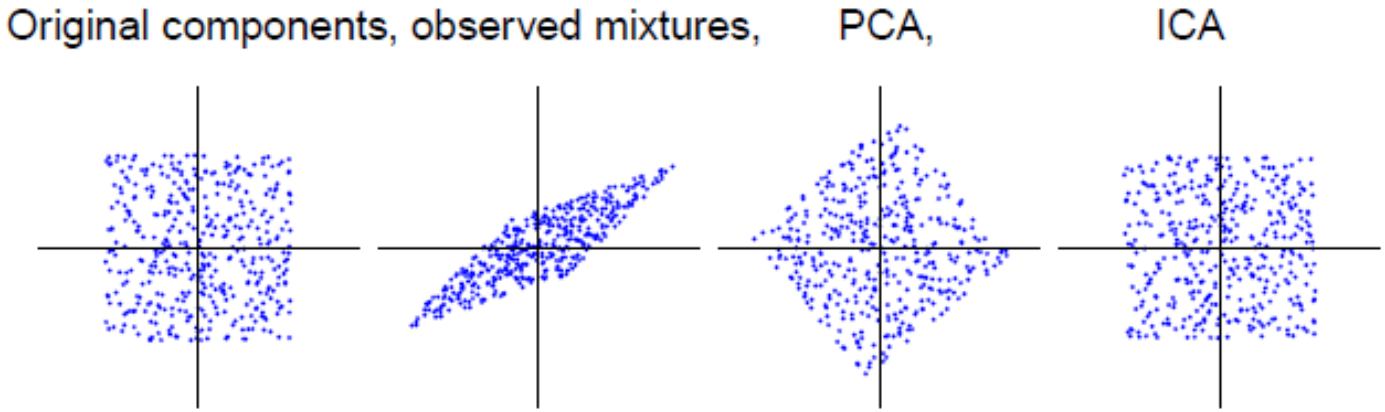
vi. Normalize $\mathrm{w} \leftarrow {}^{\mathrm{w}}/_{\|\mathrm{w}\|}$

Repeat from step-v till converge

Independent Component Analysis (ICA)

Illustration of PCA vs. ICA

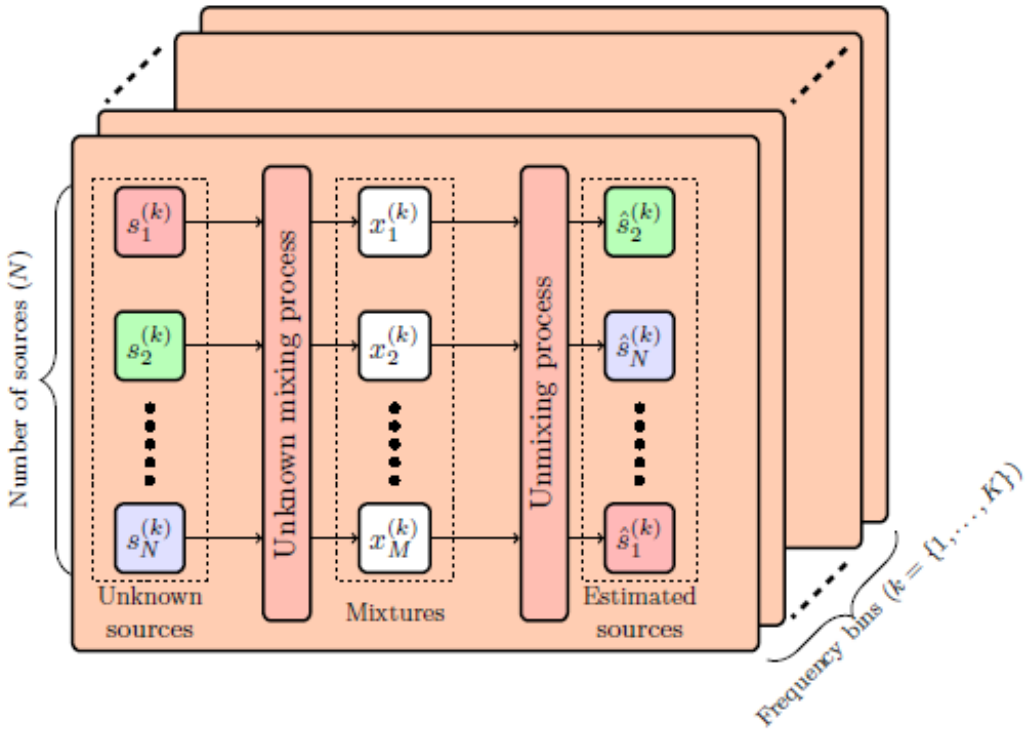
Two components with uniform distributions:



Independent Component Analysis (ICA)

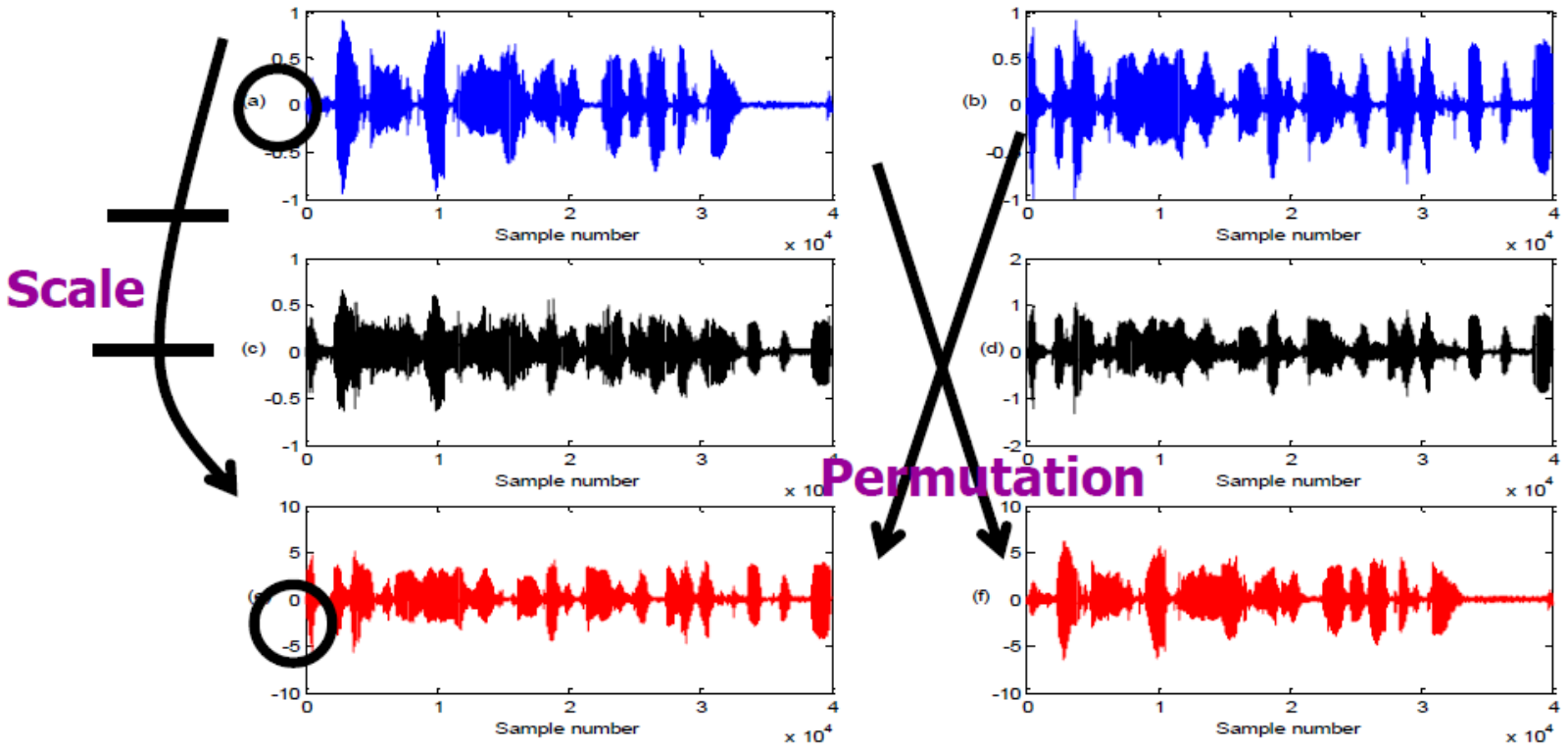
Limitations of ICA

Permutation and scaling ambiguities



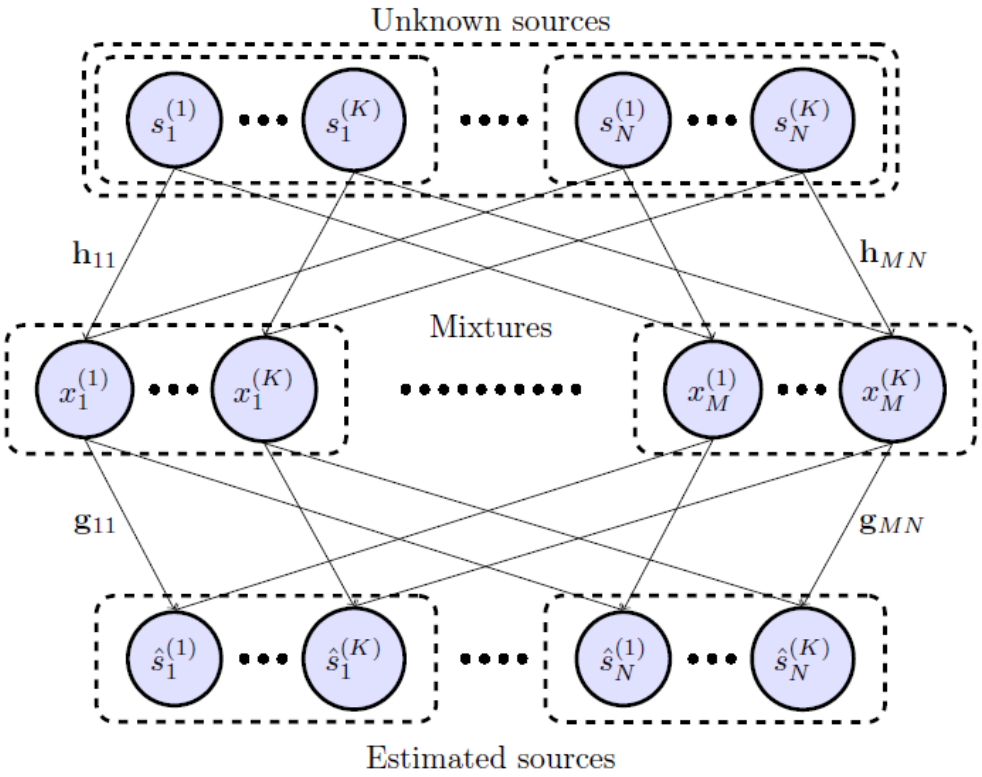
Independent Component Analysis (ICA)

Limitations of ICA



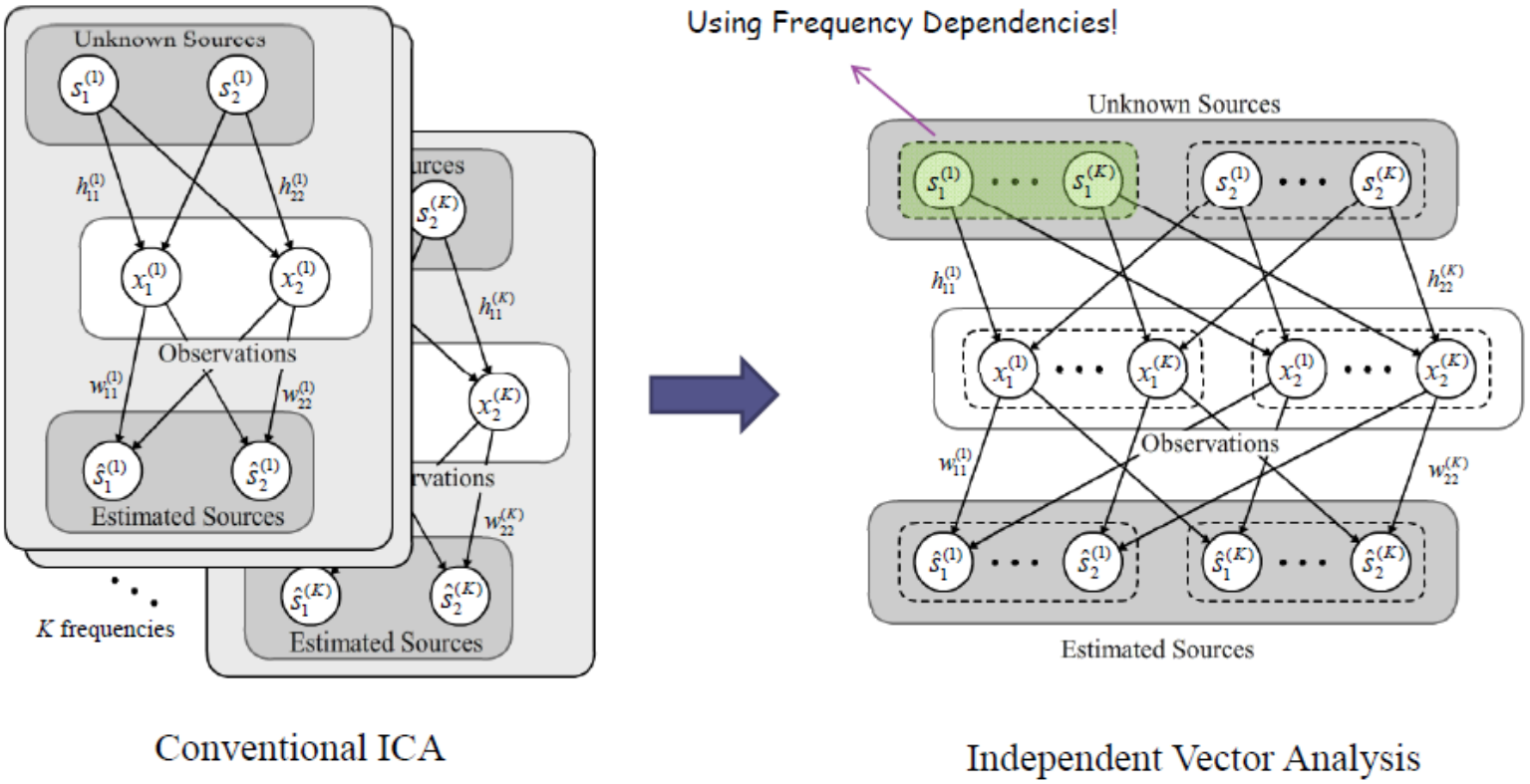
Independent Vector Analysis (IVA)

IVA models the statistical independence between sources. And maintains the dependency between frequency bins of each source



Independent Vector Analysis (IVA)

IVA vs ICA



Independent Vector Analysis (IVA)

- Modified Multivariate Cost Function

$$C = KL\left(p(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2) \,\middle\|\, \prod_{i=1}^{2} p(\hat{\mathbf{s}}_i) \right)$$

Components → Vectors !

$$\hat{\mathbf{s}}_i = \left(\hat{s}_i^{(1)}, \cdots, \hat{s}_i^{(K)} \right)^T$$

- Using a Gradient Descent Algorithm:

$$w_{ij}^{(k)new} = w_{ij}^{(k)old} + \eta \Delta w_{ij}^{(k)},$$

$$\Delta w_{ij}^{(k)} = -\frac{\partial C}{\partial w_{ij}^{(k)}} = w_{ij}^{-T(k)} - E\{\varphi^{(k)}(\hat{s}_i^{(1)} \cdots \hat{s}_i^{(K)})\} x_j^{\star(k)}$$

- Using Natural Gradient Learning:

$$\Delta w_{ij}^{(k)} = \sum_{l=1}^{2} (I_{il} - E\{\varphi^{(k)}(\hat{s}_i^{(1)} \cdots \hat{s}_i^{(K)}) \hat{s}_l^{\star(k)}\}) w_{lj}^{(k)}$$

Conventional ICA Score Function

$$\varphi(\hat{s}_i^{(k)}) = -\frac{\partial \log p(\hat{s}_i^{(k)})}{\partial \hat{s}_i^{(k)}}$$

$$= \frac{\hat{s}_i^{(k)}}{\left|\hat{s}_i^{(k)}\right|} = \exp(j \cdot \arg(\hat{s}_i^{(k)})).$$

IVA Score Function (Multivariate)

$$\varphi^{(k)}(\hat{s}_i^{(1)} \cdots \hat{s}_i^{(K)}) = \frac{\partial \sqrt{\sum_{k=1}^{K} \left|\hat{s}_i^{(k)}\right|^2}}{\partial \hat{s}_i^{(k)}} = \frac{\hat{s}_i^{(k)}}{\sqrt{\sum_{k=1}^{K} \left|\hat{s}_i^{(k)}\right|^2}}$$

Multivariate function exploiting frequency dependency!

Independent Vector Analysis (IVA)

The cost function is derived as

$$
\begin{aligned}
J_{IVA} &= \mathcal{KL}(p(\hat{\mathbf{s}}_1 \dots \hat{\mathbf{s}}_N) || \prod_{i=1}^{N} q(\hat{\mathbf{s}}_i)) \\
&= \int p(\hat{\mathbf{s}}_1 \dots \hat{\mathbf{s}}_N) \log \frac{p(\hat{\mathbf{s}}_1 \dots \hat{\mathbf{s}}_N)}{\prod_{i=1}^{N} q(\hat{\mathbf{s}}_i)} d\hat{\mathbf{s}}_1 \dots \hat{\mathbf{s}}_N \\
&= \int p(\mathbf{x}_1 \dots \mathbf{x}_M) \log p(\mathbf{x}_1 \dots \mathbf{x}_M) d\mathbf{x}_1 \dots \mathbf{x}_M \\
&- \sum_{k=1}^{K} \log |\det G^{(k)}| - \sum_{i=1}^{N} \int p(\hat{\mathbf{s}}_i) \log q(\hat{\mathbf{s}}_i) d\hat{\mathbf{s}}_i \\
&= \text{const.} - \sum_{k=1}^{K} \log |\det G^{(k)}| - \sum_{i=1}^{N} E[\log q(\hat{\mathbf{s}}_i)],
\end{aligned}
$$

Independent Vector Analysis (IVA)

Partial derivative of the cost function is employed to find gradient

$$\Delta g_{ij}^{(k)} = -\frac{\partial J_{IVA}}{\partial g_{ij}^{(k)}} = g_{ij}^{(k)-H} - E[\varphi^{(k)}(\hat{s}_i^{(1)}, \ldots, \hat{s}_i^{(K)})]x_j^{(k)*}$$

$$\text{where } \left[(G^{(k)-1})^H\right]_{ii} \equiv g_{ij}^{(k)-H}.$$

The natural gradient update becomes

$$\Delta g_{ij}^{(k)} = \sum_{l=1}^{N} (\delta_{il} - E[\varphi^{(k)}(\hat{s}_i^{(1)} \ldots \hat{s}_i^{(K)})\hat{s}_i^{(k)*}])g_{lj}^{(k)},$$

where δ_{il} is the Kronecker delta, i.e. when $i = l$,

$\delta_{il} = 1$, and zero otherwise.

Independent Vector Analysis (IVA)

Multivariate score function

$$\varphi^{(k)}(\hat{s}_i^{(1)} \dots \hat{s}_i^{(K)}) = -\frac{\partial \log q(\hat{s}_i^{(1)} \dots \hat{s}_i^{(K)})}{\partial \hat{s}_i^{(k)}}$$

A source prior applied in the original IVA

$$q(\mathbf{s}_i) \propto exp\Big(-((\mathbf{s}_i - \mu_i)^H \mathbf{\Sigma}_i^{-1}(\mathbf{s}_i - \mu_i))^{\frac{1}{2}} \Big),$$

For zero mean and unit variance data, the non-linear score function becomes

$$\varphi^{(k)}(\hat{s}_i^{(1)} \dots \hat{s}_i^{(K)}) = \frac{\hat{s}_i^{(k)}}{\sqrt{\sum_{k=1}^{K} |\hat{s}_i^{(k)}|^2}}$$

and plays important role in maintaining the dependencies between the frequency bins of each source.

Conclusions

- PCA can provide uncorrelated data and is limited to second order statistics.
- ICA is nongaussian alternative to PCA.
- ICA finds a linear decomposition by maximizing nongaussianity of the components.
- Application of ICA for convolutive source separation is constraint by the scaling and permutation ambiguities.
- IVA can mitigate the permutation problem during the learning process.
- Robustness of IVA is proportional to the strength of source priors.
- Above methods are applicable to determined and over-determined cases.

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Thank you!

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