BALANCED SENSOR MANAGEMENT ACROSS MULTIPLE TIME INSTANCES VIA L-1/L-INFINITY NORM MINIMIZATION

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ABSTRACT

In this paper, we propose a solution to the sensor management problem over multiple time instances that balances the accuracy of the sensor network estimation with its utilization. We show how this problem reduces to a binary optimization problem for which we give a convex relaxation based solution that involves the minimization of a regularized ℓ_∞ reweighted ℓ_1 norm. We show experimentally the behavior of the proposed algorithm and compare it with previous methods from the literature.

Index Terms— sensor management, convex optimization, binary optimization.

1. INTRODUCTION

The sensor management problem [1], at least in the case of a single time instance, has been extensively studied in the past. The problem is hard because deciding which sensors to select results is a binary optimization problem, that is NP-hard to solve in general. We currently identify three well-established approaches to deal with this problem. The first one is to approximate the combinatorial optimization problem by its convex relaxation [2]. This heuristic approach [3] proves to work very well in numerical experiments and to be very close to a bound on the best possible performance. Another approach, called SparSenSe [4], solves an ℓ_1 optimization problem to decide the minimum number of sensors to be activated such that an appropriately defined mean squared error (MSE) is below a given threshold. Finally, FrameSense [5] is a greedy approach that proposes to minimize the frame potential of the sensing matrix since it has been shown that this also minimizes the MSE of the estimation process.

In this paper, we study the problem of activating the minimum number of sensors from a total of m available sensors over T time instances such that required parameters are estimated with at least a minimum given accuracy at each time interval while we also balance the overall energy consumption of the sensor network. Our goal is to minimize the number of

active sensors and to balance the sensor network utilization by discouraging the selection of the same subset of sensor at each time step. The problem is interesting from a practical perspective as it has seen many applications in estimation [3], control theory [6] and wireless networks [7] for example.

Since the solution to the sensor selection problem is binary, sparsity promoting techniques [8, 9, 10] have also been applied successfully to the problem. Other approaches, like [11] uses a genetic algorithm to solve the sensor placement problem given a similar problem setup to that of [3] while [12, 13] give sub-optimal local optimization techniques and [14] proposed an exact optimization via branch-and-bound methods. While most scheduling algorithms assume that measurements are taken in a centralized manner, decentralized methods have also been proposed for the sensor management problem [4, 15].

Most previous solutions deal with the sensor management problem in a single time instance. Their solutions can be trivially extended to multiple time instances just by making the same selection at each step. This solution is unsatisfactory in most cases since it polarizes the energy consumption of the sensor network. The approaches in [16, 17, 18] were among the first to consider balancing the utilization of the sensor network by adding reweighted ℓ_1 [17] or an ℓ_2 [18] penalty to their optimization problem that encourages selection of different sensors in different time instances.

In this work, we use an ℓ_∞ regularization penalty to the optimization problem to balance the use of the sensors in the network. This natural penalty effectively translates into reducing the overall usage of any particular sensor in the network. As such, multiple selections of the same sensors are unlikely, unless absolutely necessary for estimation accuracy. This is the first paper where a combined ℓ_1/ℓ_∞ optimization problem is proposed for the sensor selection problem over multiple time instances.

2. THE SENSOR MANAGEMENT PROBLEM

In this paper, we follow, and extend, the classical setup from [3]. We assume that we want to estimate an unknown vector $\mathbf{x} \in \mathbb{R}^n$ from a set of m linear measurements given by $\mathbf{A} \in$

$\mathbb{R}^{m \times n}$ with additive white Gaussian noise $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ as $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$. The maximum-likelihood estimate is $\hat{\mathbf{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$ with the estimation error $\mathbf{e} = (\mathbf{x} - \hat{\mathbf{x}})$ having zero mean and covariance $\boldsymbol{\Sigma} = \sigma^2 \left(\mathbf{A}^T\mathbf{A}\right)^{-1}$. As discussed in [3], the logarithm of the volume of the confidence ellipsoid defined by $\boldsymbol{\Sigma}$ gives a quantitative measure of the quality of the m measurements, i.e., how informative they are about any target $\mathbf{x}$. Therefore, the convex sensor selection problem is defined in [3] as the problem of selecting a subset of $k \geq n$ sensors, where k is fixed, out of the m available such that

$$\underset{\mathbf{z};\; \mathbf{z} \in [0,1]^m,\; \mathbf{1}^T\mathbf{z}=k}{\text{maximize}} \quad \log \det\left(\mathbf{A}^T\text{diag}(\mathbf{z})\mathbf{A}\right). \qquad (1)$$

where z_i encodes if the ith sensor is activated or not. After solving (1) the resulting solution $\mathbf{z}$ might not be binary, in which case a further local search is applied to produce the final binary solution $\mathbf{z}$. Notice that this solution holds for a single time instance. We next extend the problem to multiple time instances and propose a solution for this new problem that also balances the use of the sensors.

3. THE PROPOSED METHOD

In this paper, we deal with the problem of selecting sensors out of a network of m total sensors over T time instances. At each time instant we guarantee that a minimum level of accuracy is met and overall the network usage is balanced. We propose a ℓ_∞ regularized reweighted ℓ_1 optimization procedure to deal with this problem.

We start by introducing the matrix $\mathbf{Z} = \begin{bmatrix} \mathbf{z}_1 & \dots & \mathbf{z}_T \end{bmatrix} \in \{0,1\}^{m \times T}$, where each $\mathbf{z}_t \in \{0,1\}^m$ is a binary variable that decides which sensors out of the m available are selected at time instant t with $t = 1, \dots, T$. In order to balance the usage of the sensors (and avoid the consecutive selection of the same subset of sensors) we introduce the utilization factor

$$\mathbf{u} = \sum_{t=1}^{T} \mathbf{z}_t \in \{0, 1, \dots, T\}^{m \times 1}, \qquad (2)$$

which is just the sum over the selection variables $\mathbf{z}_t$.

Ideally, for a fixed regularization parameter $\lambda \geq 0$, we would like to solve exactly the optimization problem

$$\underset{\mathbf{Z} \in \{0,1\}^{m \times T}}{\text{minimize}} \quad \sum_{t=1}^{T} \|\mathbf{z}_t\|_0 + \lambda \|\mathbf{u}\|_\infty$$

$$\text{subject to} \quad \log \det(\mathbf{A}^T\text{diag}(\mathbf{z}_t)\mathbf{A}) \geq \mu_t,\; t = 1, \dots, T, \qquad (3)$$

where $\|\mathbf{z}\|_0$ is the ℓ_0 pseudo-norm, i.e., the number of non-zero entries, and $\|\mathbf{z}\|_\infty = \max_{1 \leq i \leq m} |z_i|$ is the ℓ_∞ norm. Because all the entries of $\mathbf{Z}$ are positive, $\|\mathbf{u}\|_\infty$ can be viewed as the ℓ_∞ norm induced on $\mathbb{R}^m$ and $\mathbb{R}^T$, also called the max-row-sum norm, given by $\|\mathbf{Z}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{T} |Z_{ij}|$.

Algorithm 1 – Sensor scheduling by ℓ_1/ℓ_∞ minimization.
Input: The design matrix with m sensors $\mathbf{A} \in \mathbb{R}^{m \times n}$, the regularization parameter $\lambda > 0$, the number of time instances T and the minimum accuracy at each time instant μ_t.
Output: The scheduling table $\mathbf{Z} \in \{0,1\}^{m \times T}$ for the sensor activations at each time step.

Initialization:
   1. Set initial weights $\mathbf{w}_t = \mathbf{1}, t = 1, \dots, T$ and initial all-zero solution $\mathbf{Z}_{\text{prev}} = \mathbf{0}$.
   2. Initialize sets $\mathcal{N} = \emptyset$ and $\mathcal{K} = \emptyset$.

Iterations:
   1. Solve (8) to obtain current estimate $\mathbf{Z}$.
   2. Update the sets $\mathcal{N} = \{n \mid \mathbf{Z}(n) \leq \epsilon\}$ and $\mathcal{K} = \{k \mid \mathbf{Z}(k) \geq 1 - \epsilon\}$.
   3. If iterative process has converged, i.e., $\|\mathbf{Z} - \mathbf{Z}_{\text{prev}}\|_F^2 \leq \epsilon$, then $\mathcal{K} = \mathcal{K} \cup \{\arg\max_k \mathbf{Z}(k), k \notin \mathcal{K}\}$.
   4. If solution is binary, i.e., $|\mathcal{N}| + |\mathcal{K}| = mT$, then stop.
   5. Update weights according to (6) and store current solution in $\mathbf{Z}_{\text{prev}}$.

According to the objective function of (3), for a given time instance we encourage the selection of only a subset of the sensors while overall we discourage the selection of the same sensors in all time instances (except if the constraints are not otherwise met). The constraints guarantee that in all time instances we get a minimum level of accuracy μ_t from the measurements taken.

An important question is how to choose the levels μ_t. The objective function of (1) is maximized when all measurements are used, i.e., $\mathbf{z} = \mathbf{1}$. To see this, assume $k \geq n$ measurements have been already selected by the activation vector $\mathbf{z}$ in $\mathbf{X} = \mathbf{A}^T\text{diag}(\mathbf{z})\mathbf{A}$ and a new measurement $\mathbf{a}_j$ has been chosen (i.e., set $z_j = 1$), by the matrix determinant lemma we have that $\det(\mathbf{X} + \mathbf{a}_j\mathbf{a}_j^T) = (1 + \mathbf{a}_j^T\mathbf{X}^{-1}\mathbf{a}_j)\det(\mathbf{X}) > \det(\mathbf{X})$, where the last inequality is true since $\mathbf{X}$ is full rank and positive definite. We therefore choose μ_t to be fractions of this maximum level

$$\mu_t = \rho_t \log \det(\mathbf{A}^T\text{diag}(\mathbf{1})\mathbf{A}),\; \rho_t \in [0,1). \qquad (4)$$

We consider that allowing the user to select the variables μ_t is a realistic scenario. The user needs to input the desired accuracy of the measurements, at least as a measure of the maximum accuracy possible with the full given sensor network. Ultimately, this choice decides the number of sensors selected across the T time instances.

Still, the optimization problem (3) is not convex. We proceed now to relax it to a convex expression and propose an iterative way to solve it. First, the binary constraint is relaxed to $\mathbf{Z} \in [0,1]^{m \times T}$ and the ℓ_0 norm is replaced by the convex ℓ_1 norm $\|\mathbf{z}\|_1 = \sum_{i=1}^{m} |z_i|$. Notice that $\|\mathbf{z}\|_1 = \mathbf{1}^T\mathbf{z}$ because all entries of $\mathbf{z}$ are positive. For the same reason a simplification

also occurs for the ℓ_∞ norm that is now replaced by a maximum operation, i.e., $\|\mathbf{z}\|_\infty = \max\limits_{1 \leq i \leq m} z_i$. Furthermore, we will also solve the new optimization problem in an iterative fashion to ensure that the final result $\mathbf{Z}$ is binary. Therefore, we choose to follow the iterative reweighted ℓ_1 [19, 20] approach for improved results. Classically, this means replacing the $\|\mathbf{z}\|_1$ objective with $\|\mathbf{Wz}\|_1$ where $\mathbf{W}$ is a diagonal matrix containing the weights. In our case, this objective simplifies to $\mathbf{w}_t^T \mathbf{z}_t$. The choice of the weight vectors $\mathbf{w}_t \in \mathbb{R}^m$ is discussed later. Finally, the convex problem we reach is

$$\underset{\mathbf{Z} \in [0,1]^{m \times T}}{\text{minimize}} \quad \sum_{t=1}^T \mathbf{w}_t^T \mathbf{z}_t + \lambda \max(\mathbf{u})$$

$$\text{subject to} \quad \log \det(\mathbf{A}^T \text{diag}(\mathbf{z}_t)\mathbf{A}) \geq \mu_t, \ t = 1, \ldots, T. \tag{5}$$

We will deal with this problem in an iterative fashion, updating the weights until convergence. The optimization problem above is solved via the CVX generic solver [21]. Details of the procedure and modifications to (5) are discussed now.

In solving (5), the choice of the weights $\mathbf{w}_t$ is very important. Initially, the weights are set to $\mathbf{w}_t = \mathbf{1}$ (assuming no prior information about the solution is available, all activations are equally likely to be zero or one). After solving (5) for the first time we obtain a solution $\mathbf{Z}$ and the weights are set inversely proportional to the magnitude of the entries, i.e., a well known choice[19] is $\mathbf{w}_t = \mathbf{1} \oslash (|\mathbf{z}_t| + \epsilon)$, where $\oslash$ is the elementwise division operator and ϵ is a small positive constant. In this paper, we choose to update the weights in a different manner, taking advantage of the fact that ultimately the solution needs to be binary. We propose to use $\mathbf{w}_t = \mathbf{1} - \frac{\mathbf{z}_t}{\|\mathbf{z}_t\|_\infty}$, and since with high probability at least one entry in $\mathbf{z}_t$ will achieve the maximum value of one we can further simplify the weights to

$$\mathbf{w}_t = \mathbf{1} - \mathbf{z}_t. \tag{6}$$

To motivate our choice for $\mathbf{w}_t$ let us analyze what happens after solving (5) for a certain number of iterations and the optimization process converges. The stationary point of the first term in the objective function is

$$\mathbf{w}_t^T \mathbf{z}_t = (\mathbf{1} - \mathbf{z}_t)^T \mathbf{z}_t = \mathbf{1}^T \mathbf{z}_t - \mathbf{z}_t^T \mathbf{z}_t = k_t - \|\mathbf{z}_t\|_2^2, \tag{7}$$

where k_t is the number of sensors active at time instant t. Seen this way, our heuristic is a method for ℓ_2 norm maximization. This is a problem known to be hard [22] but very useful in combinatorial optimization since the ℓ_2 objective is maximized exactly when the solution $\mathbf{z}_t$ is binary. Also, notice that as the entries of $\mathbf{Z}$ are pushed to binary values the whole objective function in (5) reduces to $\lambda \max(\mathbf{u})$. We also introduce an additional constraint that every sensor needs to be selected at least once over all time instances, i.e., $\mathbf{u} \geq \mathbf{1}$.

Unfortunately, a last technical difficulty of our proposed method is that, no matter the choice of the weights, we cannot guarantee that $\mathbf{Z}$ will converge to a binary solution. Previous works [3, 18] apply a local search procedure after the
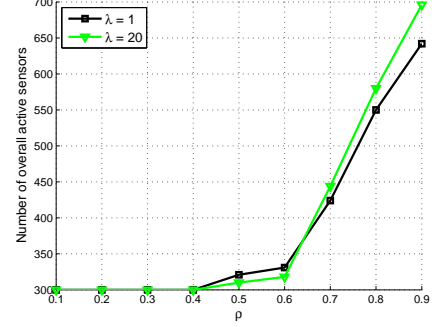

Fig. 1: The total number of sensor activations over all time instances scheduled by Algorithm 1 for different values of the regularization parameter λ. We vary the accuracy of the estimation as a fraction of the best accuracy achievable by the full sensor network (4) with $m = 100$ and $n = 30$. Results are averaged over 50 realizations of the sensor network.

main optimization has converged to finally drive the solution to have binary entries. In this paper, we take a slightly different approach. During the entire iterative process we will keep track of two sets $\mathcal{N}$ and $\mathcal{K}$, containing the coordinates of the zero and the one entries, respectively, in $\mathbf{Z}$. Practically, we will test if entries are below a threshold ϵ or above a threshold $1 - \epsilon$, respectively. All entries indexed in this set are no longer considered to be variables in the optimization process but to have become fixed, final. This assumption is realistic since entries that are very close to zero or one have a very low probability of switching their values due to the weights put in place by $\mathbf{w}_t$. After the iterative process converges, the entry of $\mathbf{Z}$ with the highest value that is not indexed in $\mathcal{K}$ is added to this set, i.e., it is set to one, and the optimization process continues with the other variables still to be decided. The whole procedure stops when every entry of $\mathbf{Z}$ is either in $\mathcal{N}$ or in $\mathcal{K}$, i.e., $|\mathcal{N}| + |\mathcal{K}| = mT$. Notice that using these two sets we also reduce the size of the optimization problems that are solved and thus we speed-up solving (5) as the algorithm progresses.

Therefore, the final optimization problem we solve is

$$\underset{\mathbf{Z} \in [0,1]^{m \times T}}{\text{minimize}} \quad \sum_{t=1}^T \mathbf{w}_t^T \mathbf{z}_t + \lambda \max(\mathbf{u})$$

$$\text{subject to} \quad \mathbf{u} \geq \mathbf{1}, \ \mathbf{Z}(n) = 0, \forall n \in \mathcal{N}, \ \mathbf{Z}(k) = 1, \forall k \in \mathcal{K}$$

$$\log \det(\mathbf{A}^T \text{diag}(\mathbf{z}_t)\mathbf{A}) \geq \mu_t, \ t = 1, \ldots, T. \tag{8}$$

The full proposed procedure is described in Algorithm 1.

4. EXPERIMENTAL RESULTS

In this section, we give numerical results that show sensor selections calculated by our proposed method, Algorithm 1. We follow a setup similar to the one in [3]. The measurement matrix $\mathbf{A}$ is chosen randomly and independently from a $\mathcal{N}(0,1)$
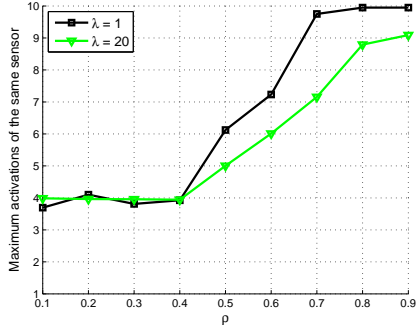
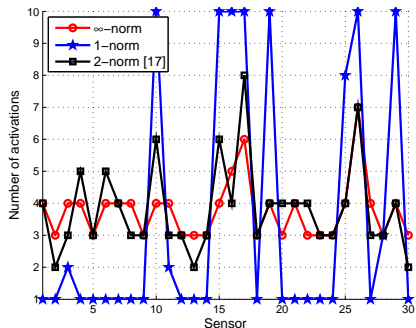**Fig. 2**: The maximum number of activations of any single sensor. Experimental setup similar to Fig. 1.

Fig. 3: We compare our ℓ_∞ regularization approach to previously proposed regularization based on the ℓ_2 norm. We also show the effects of ℓ_1 regularization. Experimental setup is $\lambda = 10$, $\rho = 0.6$, $m = 30$ and $n = 10$.

distribution. We evaluate the sensor management solution over $T = 10$ time instances for different values of μ_t. We ask for the same accuracy of the estimation at each time instant and therefore we fix $\mu_t = \mu = \rho \log \det(\mathbf{A}^T \mathrm{diag}(\mathbf{1})\mathbf{A})$.

In Figs. 1 and 2 we show how sensor activations are scheduled by the proposed Algorithm 1. We consider an example where $n = 30$ parameters are estimated by a network of $m = 100$ sensors. Fig. 1 shows the total number of sensor activations over all time instances as a function of the minimum required estimation accuracy given by ρ. Notice that increasing the regularization parameter λ also increases the total number of active sensors in the network. On the other hand, Figure 2 shows the upside of a large value for the regularization parameter: a single sensor is selected fewer times over all time instances balancing therefore energy consumption and network utilization. This is the trade-off that an ℓ_1/ℓ_∞ objective function of (8) outlines. Both the total number of sensor activations and the maximum utilization of any sensor increase as the accuracy of estimation ρ increases. Demanding high estimation accuracy (values $\rho \geq 0.7$) leads to situations where inevitably at least one sensor is selected in almost all time instances.

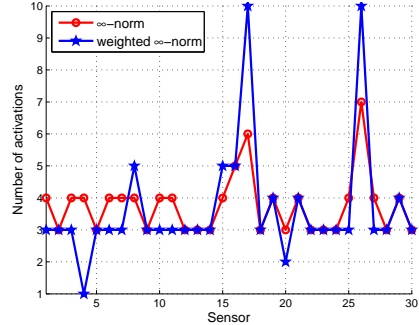In Fig. 3 we show the sensor selection frequency for one

Fig. 4: Weighted versus un-weighted ℓ_∞ regularization. Experimental setup the same as Fig. 3.

run of Algorithm 1. For simplicity of exposition we now consider an example where $n = 10$ parameters are estimated by a network of $m = 30$ sensors. We also test against the approach in [18] that uses a ℓ_2 penalty to regularize the problem. Note that [18] has a different experimental setup but we can still use the regularization idea and apply it in our scenario. The ℓ_∞ proposed penalty leads to a situation where each sensor is selected on average 3.76 times with standard deviation 0.9352 while in the ℓ_2 approach selects more sensors with average 3.83 and standard deviation 1.42.

Our framework also allows an easy way to encode prior information about the energy constraints of any sensor. Assume for example that there is a subset of sensors in the network that have a virtually unlimited power supply or that some sensors have more power available than others. We can alter the objective function of (8) to

$$\sum_{t=1}^{T} \mathbf{w}_t^T \mathbf{z}_t + \lambda \max(\mathbf{W}\mathbf{u}), \qquad (9)$$

where the diagonal matrix $\mathbf{W}$ encodes information about the power supplies of the sensors. For example, $w_{ii} = 0$ encodes that sensor i has a large power supply and therefore there is no penalty of selecting it multiple times. Fig. 4 compares (9) where $\mathbf{W} = \mathbf{I}$ with (9) where now $w_{ii} = 1$ except for $w_{4,4} = w_{20,20} = 2$ and $w_{17,17} = w_{26,26} = 0$. Notice that sensors 17 and 26 are used in all time instances while the use of sensors 4 and 20 is reduced in the weighted ℓ_∞ approach.

5. CONCLUSIONS

In this paper, we propose a novel algorithm based on ℓ_∞ regularized reweighted ℓ_1 optimization to build activation schedules for a network of sensors so that we are able to estimate parameters with a given accuracy while we also balance the energy utilization of the network. We show the flexibility of our method when it comes to incorporating power consumption information into the sensor management problem and show its superiority to previous methods in the literature that proposed different regularization.

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