Introduction to Array Processing

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UDRC-EURASIP Summer School, Edinburgh, 30 June 2022

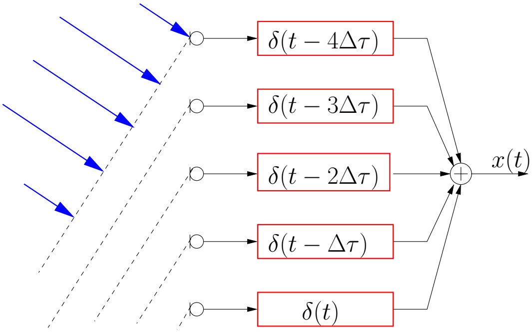
This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) Grant number EP/S000631/1 and the MOD University Defence Research Collaboration in Signal Processing.

Introduction to Array Processing — Overview

Contents:

1.1 Intuitive Beamforming

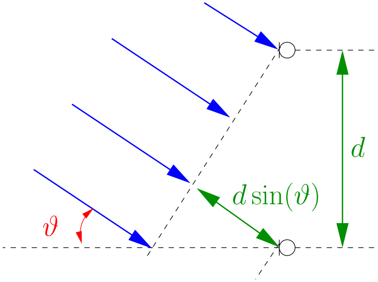
- A farfield wavefront arrives at a sensor array:



- due to the direction of arrival (DOA) and finite propagation speed, the wavefront will arrive at different sensors with a delay $\Delta\tau$;
- with appropriate processing (beamforming), the sensor signals can be aligned to create constructive interference at the output $x(t)$.

1.2 Spatial Sampling

- For unambiguous spatial sampling, we need to take at least two samples per wavelength of the highest frequency component in the array signals;
- analogy from temporal sampling (Nyquist): take at least two samples per period (relating to the highest frequency component);
- Wavelength λ and frequency f are related by the propagation speed c in the medium: $\lambda = \frac{c}{f}$;



- maximum sensor distance

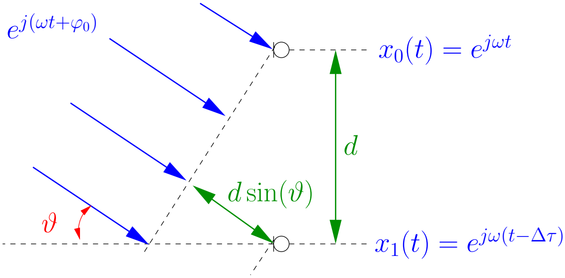
$$d = \frac{\lambda_{\max}}{2} = \frac{c}{2 f_{\max}}$$

- time delay between sensors

$$\Delta\tau = \frac{d \sin(\vartheta)}{c} = \frac{\sin(\vartheta)}{2 f_{\max}}$$

Spatial and Temporal Sampling

- Consider the array signals $x_0(t)$ and $x_1(t)$ due to a source $e^{j(\omega t+\varphi_0)}$:



- sampling with $t = nT_\mathrm{s}$ leads to

$$x_0[n] = e^{j\omega nT_\mathrm{s}} \qquad \text{and} \qquad x_1[n] = e^{j\omega(nT_\mathrm{s}-\Delta\tau)}$$

- with $f_\mathrm{max} = \frac{f_\mathrm{s}}{2} = \frac{1}{2T_\mathrm{s}}$ and normalised angular frequency $\Omega = \omega T_\mathrm{s}$,

$$x_0[n] = e^{j\Omega n} \qquad \text{and} \qquad x_1[n] = e^{j\Omega n} \cdot e^{-j\Omega\sin(\vartheta)}$$

1.3 Steering Vector

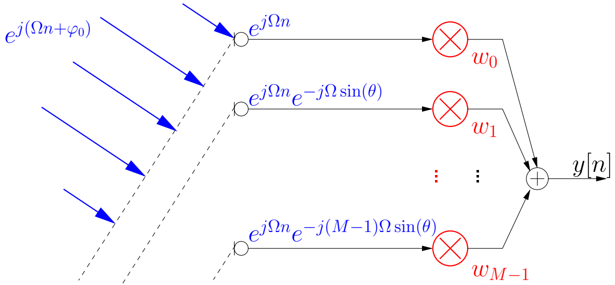
- A narrowband source with norm. angular frequency Ω illuminates an M-element linear array of equi-spaced sensors:

$$\mathbf{x}[n] = \begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{M-1}[n] \end{bmatrix} = e^{j\Omega n} \cdot \begin{bmatrix} 1 \\ e^{-j\Omega \sin(\vartheta)} \\ \vdots \\ e^{-j(M-1)\Omega \sin(\vartheta)} \end{bmatrix} = e^{j\Omega n} \cdot \mathbf{s}_{\Omega,\vartheta}$$

- the vector $\mathbf{s}_{\Omega,\vartheta}$ characterises the phase shifts of waveform with frequency Ω and DOA ϑ measured at the array sensors;
- since a narrowband signal $e^{j\Omega n}$ only causes phase shifts rather than delays, constructive interference can be accomplished by a set of complex multipliers rather than processors $\delta(t - m\Delta\tau)$, $m = 0, 1, \ldots (M-1)$;
- beamforming problem: how to select the set of complex coefficients?

1.4 Data Independent Beamformer

- Find a set of complex multipliers w_m, $m = 0, 1, \ldots (M-1)$:



- to steer the array characteristic towards this source, the output

$$y[n] = [w_0 \;\; w_1 \;\; \ldots \;\; w_{M-1}] e^{j\Omega n} \begin{bmatrix} 1 \\ e^{-j\Omega \sin(\vartheta)} \\ \vdots \\ e^{-j(M-1)\Omega \sin(\vartheta)} \end{bmatrix} = e^{j\Omega n} \mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\vartheta}$$

should satisfy $y[n] = e^{j\Omega n}$, leading to $\mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\vartheta} = 1$.

Coefficient Vector

- For later convenience and compatibility, the Hermitian transpose operator $\{\cdot\}^{\mathrm{H}}$ is used to denote the coefficient vector

$$\mathbf{w}^{\mathrm{H}} = [w_0 \;\; w_1 \;\; \ldots \;\; w_{M-1}]$$

- as a result, the vector $\mathbf{w}$ hold the complex conjugates of the coefficients,

$$\mathbf{w} = \begin{bmatrix} w_0^* \\ w_1^* \\ \vdots \\ w_{M-1}^* \end{bmatrix}$$

- to access the actual unconjugated coefficients, the beamforming vector $\mathbf{w}^*$ has to be considered
- note that

$$\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega,\vartheta} = 1 \qquad \longrightarrow \qquad \mathbf{s}^{\mathrm{H}}_{\Omega,\vartheta}\mathbf{w} = 1$$

Narrowband Beamforming — Single Source

- The expression $\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}\mathbf{w} = 1$ forms a system with one single equation and M unknowns

$$\boxed{\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}}\;\boxed{\mathbf{w}} = \boxed{1}$$

- general solution to an underdetermined system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is the right pseudo-inverse $\mathbf{A}^{\dagger}$,

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} = \mathbf{A}^{\mathrm{H}}(\mathbf{A}\mathbf{A}^{\mathrm{H}})^{-1}\mathbf{b}$$

- here:

$$\mathbf{w} = (\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}})^{\dagger} \cdot 1 = \mathbf{s}_{\Omega,\vartheta} \cdot (\mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}}\mathbf{s}_{\Omega,\vartheta})^{-1} \cdot 1 = \frac{\mathbf{s}_{\Omega,\vartheta}}{\|\mathbf{s}_{\Omega,\vartheta}\|_2^2} = \frac{1}{M}\mathbf{s}_{\Omega,\vartheta}$$

- the complex conjugation for $\mathbf{w}^*$ inverts and therefore compensates the phase of the steering vector, which could have been foreseen
- the formulation via the pseudo-inverse will be powerful for more complicated cases.

Narrowband Beamformer Example

- Source parameters: $\Omega = \frac{\pi}{2}$ and $\vartheta = 30^\circ$; array parameter: $M = 5$;
- steering vector (with $\Omega \sin(\vartheta) = \frac{1}{4}\pi$):

$$\mathbf{s}^{\mathrm{T}}_{\Omega,\vartheta} = [1 \;\; e^{-j\frac{1}{4}\pi} \quad \ldots \quad e^{-j\frac{4}{4}\pi}]$$

- coefficient vector is given by $\mathbf{w} = (\mathbf{s}^{\mathrm{H}}_{\Omega,\vartheta})^{\dagger}$;
- numerical solution in Matlab;

```
Omega=1/4; theta = pi/6; M=5;
s = exp(-sqrt(-1)*Omega*sin(theta)*(0:(M-1)'));
w = pinv(s');
```

- `angle([s conj(w)])/pi` yields:

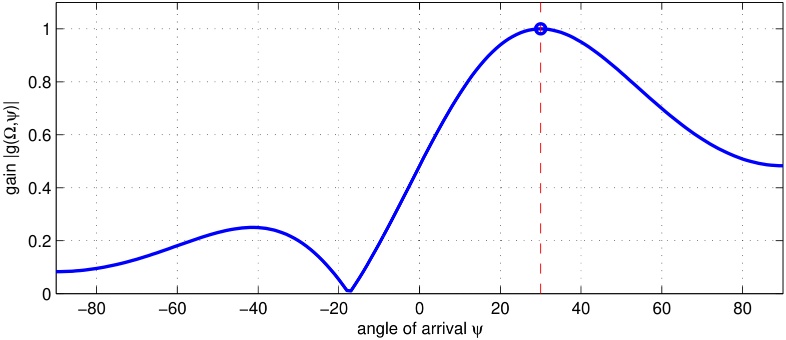
```
-0.00000   0.00000
-0.25000   0.25000
-0.50000   0.50000
-0.75000   0.75000
-1.00000   1.00000
```

Beam Pattern I

- The beamformer has a unit gain towards a source with frequency Ω and DoA θ; what is its gain response towards other angles of arrival?
- the beam pattern measures the response of a beamformer by sweeping the angle ψ of a source with frequency Ω

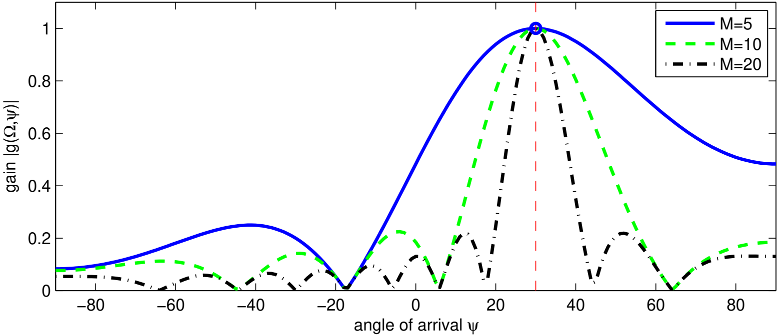
$$g(\Omega,\psi) = \mathbf{w}^{\mathrm{H}} \mathbf{s}_{\Omega,\psi}$$

- beam pattern for the previous example:



Beam Pattern II

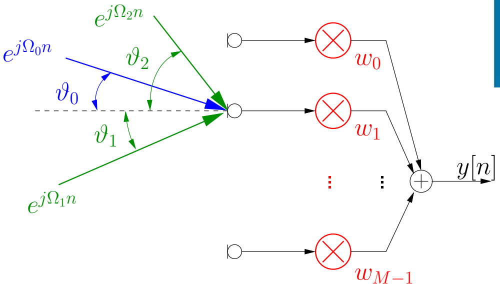
- Below are a number of beam patterns for the case $\Omega = \frac{\pi}{2}$ and $\vartheta = 30°$ for variable M;



- increasing the sensor number M narrows the main beam, and increases the number of spatial zeros;
- analogous characteristic in the time domain: increased temporal support leads to higher frequency resolution.

Interference

- Many scenarios contain a source of interest and a number of interferers:
  signal of interest:
  $\{\Omega_0, \vartheta_0\}$
  two interferers:
  $\{\Omega_1, \vartheta_1\}$, $\{\Omega_2, \vartheta_2\}$



- we would like to control the beampattern to place spatial nulls in the directions of the interfering sources;
- Problem formulation and solution :

$$\begin{bmatrix} \mathbf{s}^{\mathrm{H}}_{\Omega_0,\vartheta_0} \\ \mathbf{s}^{\mathrm{H}}_{\Omega_1,\vartheta_1} \\ \mathbf{s}^{\mathrm{H}}_{\Omega_2,\vartheta_2} \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \longrightarrow \qquad \mathbf{w} = \begin{bmatrix} \mathbf{s}^{\mathrm{H}}_{\Omega_0,\vartheta_0} \\ \mathbf{s}^{\mathrm{H}}_{\Omega_1,\vartheta_1} \\ \mathbf{s}^{\mathrm{H}}_{\Omega_2,\vartheta_2} \end{bmatrix}^{\dagger} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Narrowband BF Example — Multiple Sources

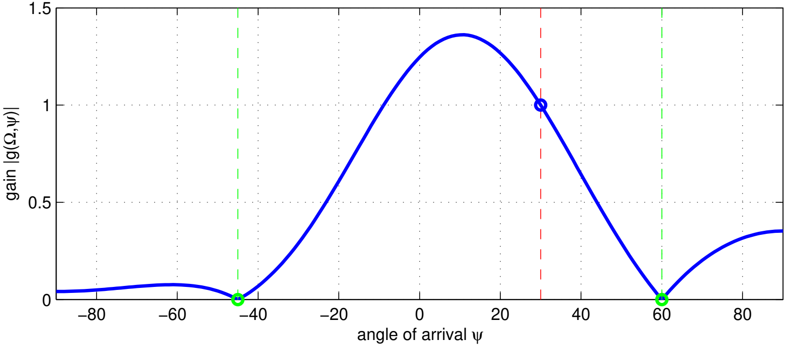
- The signal of interest illuminates an $M = 5$ element array at a frequency $\Omega_0 = \frac{\pi}{2}$ with a DoA $\vartheta_0 = 30°$
- two interferers at $\Omega_1 = \Omega_2 = \Omega_0$ are present with DoA $\vartheta_1 = -45°$ and $\vartheta_2 = 60°$
- results via right pseudo-inverse of steering vectors

$\angle \mathbf{s}_{\Omega_0,\vartheta_0}$	$\angle \mathbf{s}_{\Omega_1,\vartheta_1}$	$\angle \mathbf{s}_{\Omega_2,\vartheta_2}$	$\angle \mathbf{w}^*$	$\lvert\mathbf{w}\rvert$
0.00	0.00	0.00	-42.81	0.3172
45.00	63.64	-77.94	-105.01	0.3004
90.00	127.28	-155.89	-90.00	0.2343
135.00	-169.08	126.17	-74.99	0.3004
180.00	-105.44	48.23	-137.19	0.3172

- the angle of $\mathbf{w}$ is no longer intuitive; also note that the coefficients in $\mathbf{w}$ no longer have the same modulus
- amongst a manifold of solutions, the right pseudo-inverse provides the minimum norm solution.

Multiple Source Example — Beampattern

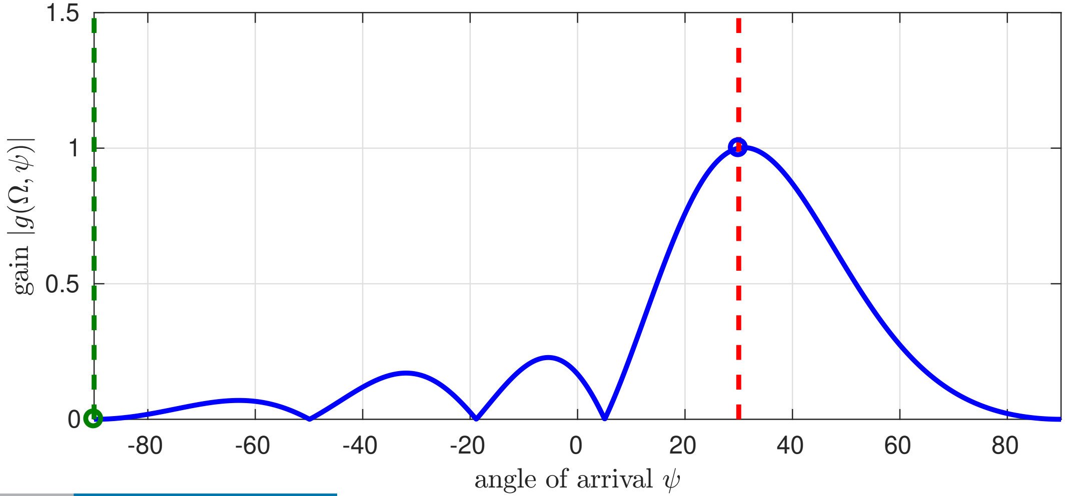
- Beam pattern one source of interest and two interferers:



- the pseudo-inverse is the minimum norm solution, keeping the general gain response as low as possible;
- the minimum norm property protects against spatially white noise.

Beamforming Example — Variable Interferer I

- $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:



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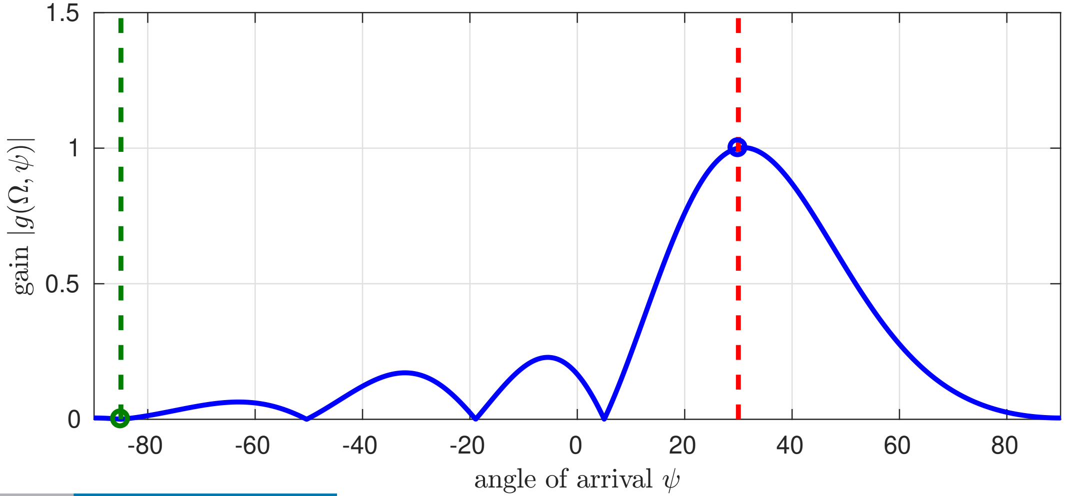

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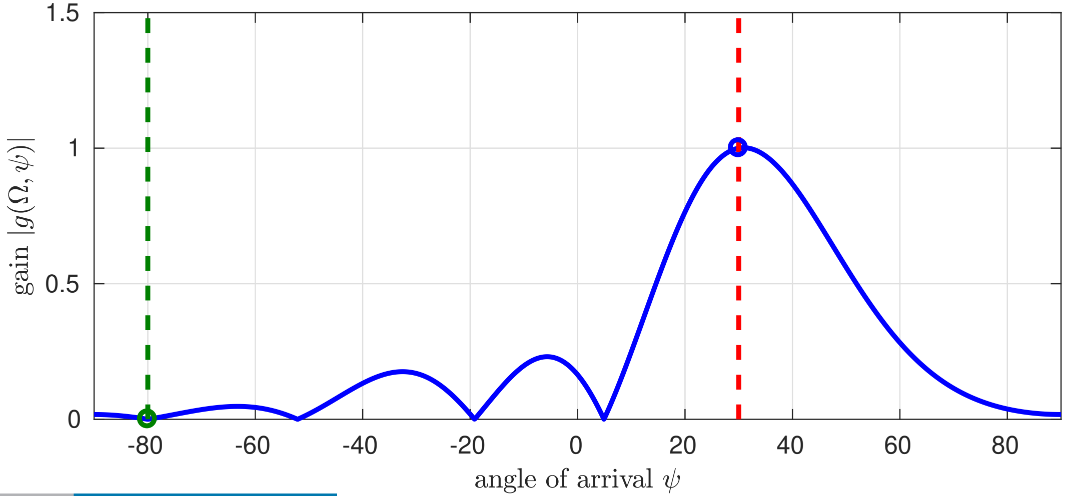

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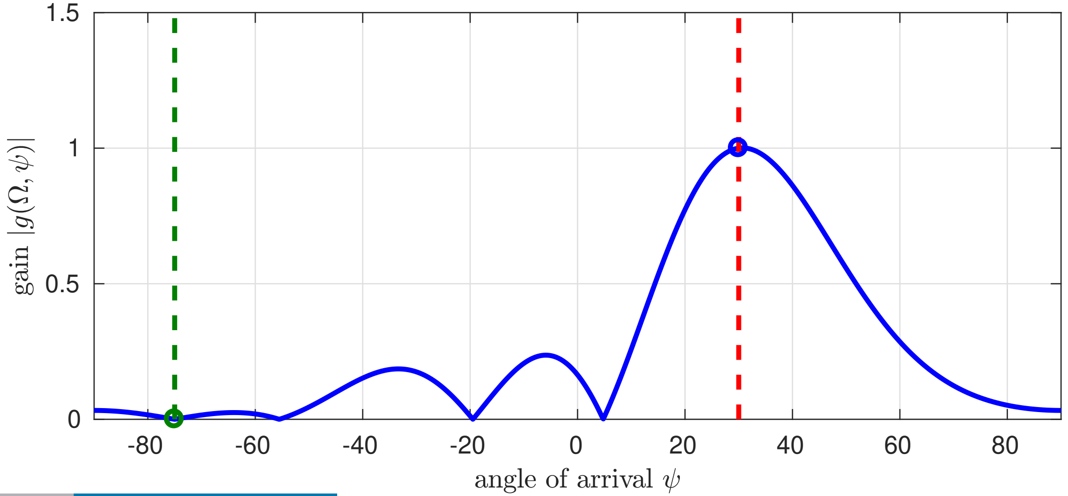

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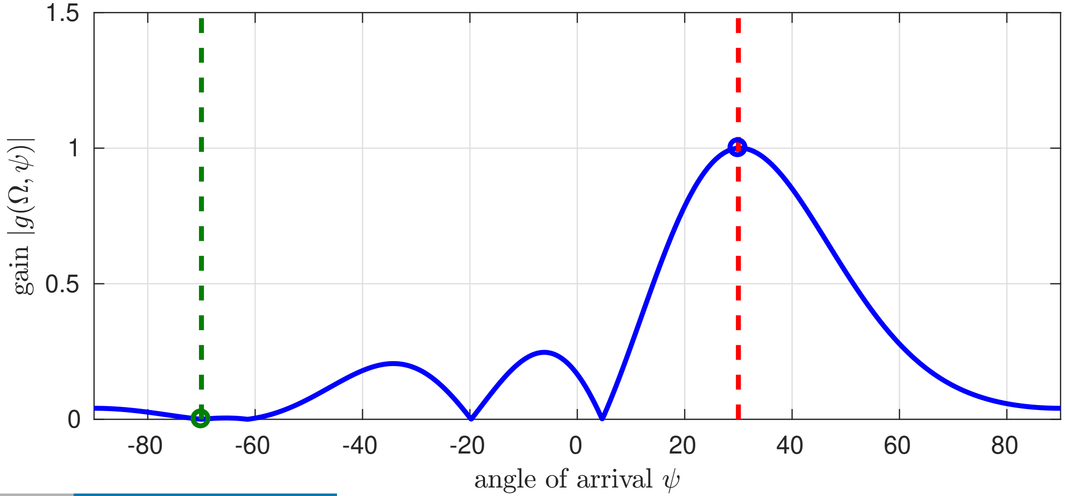

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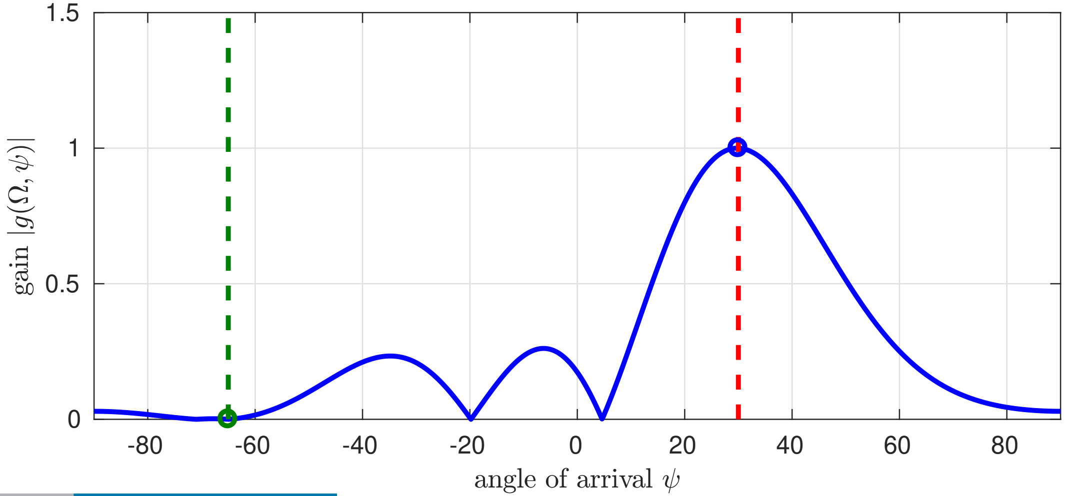

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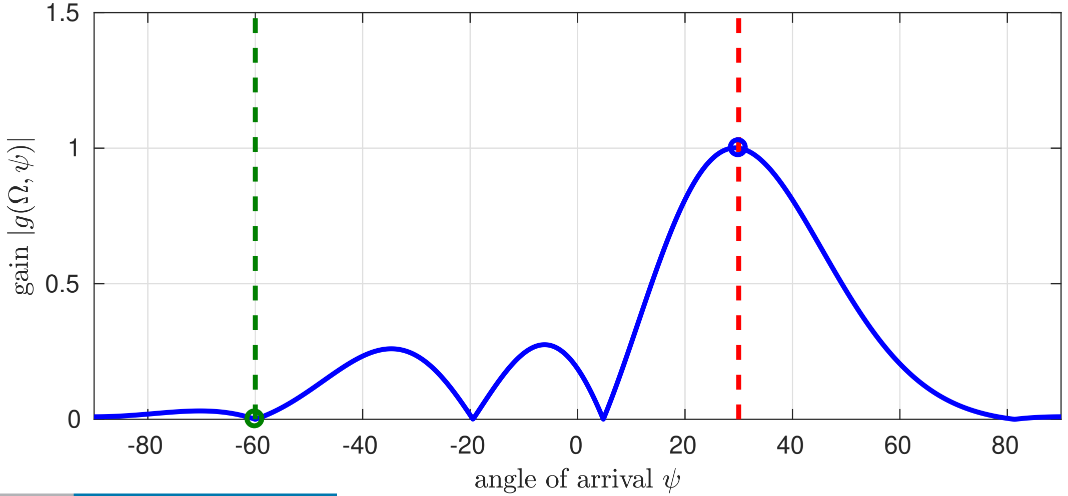

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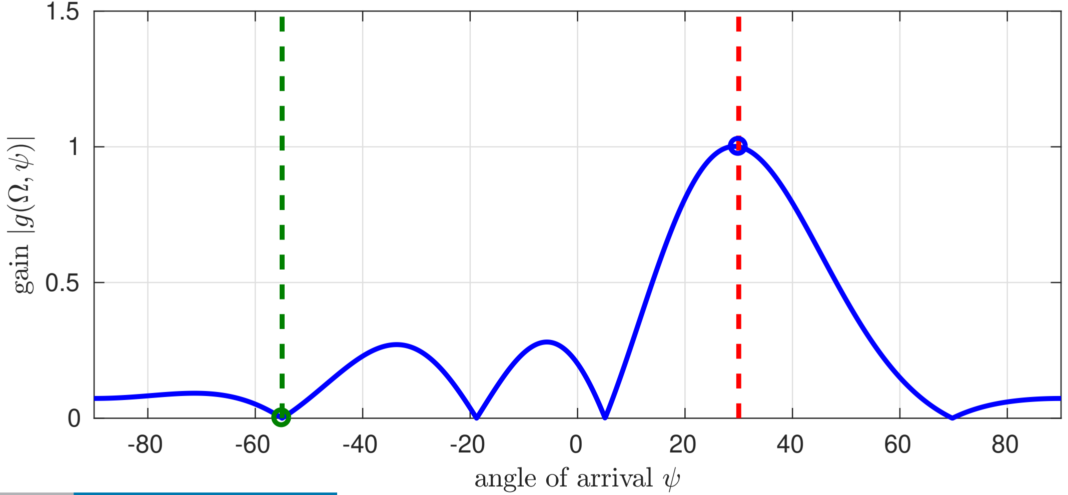

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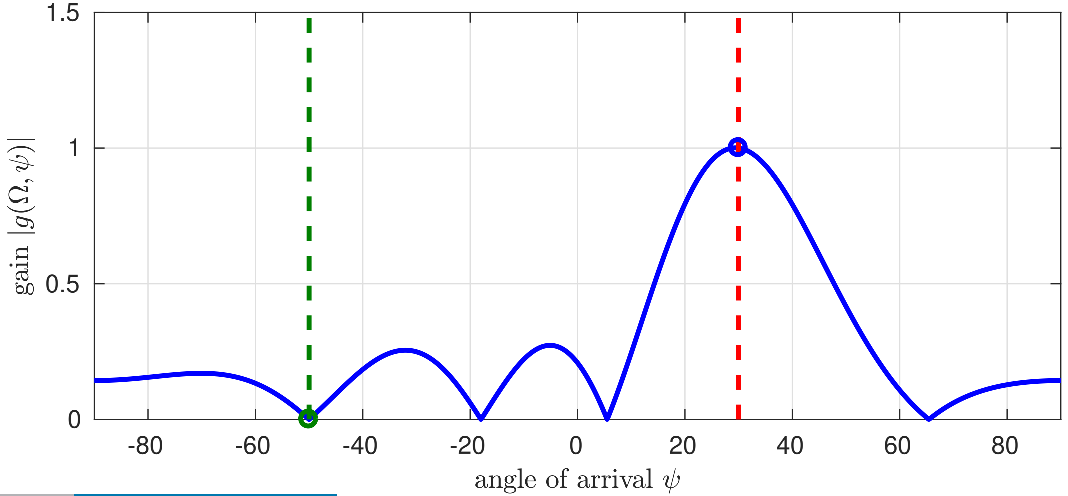

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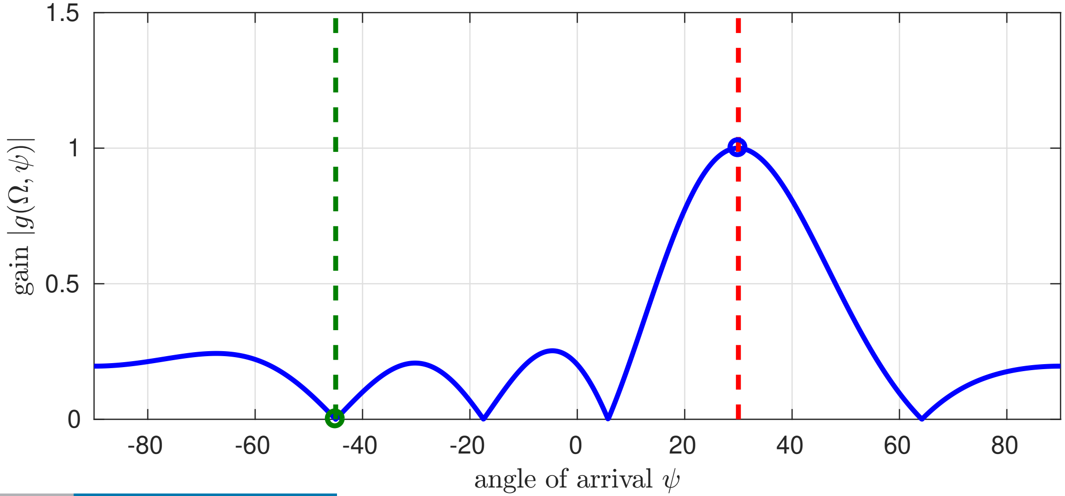

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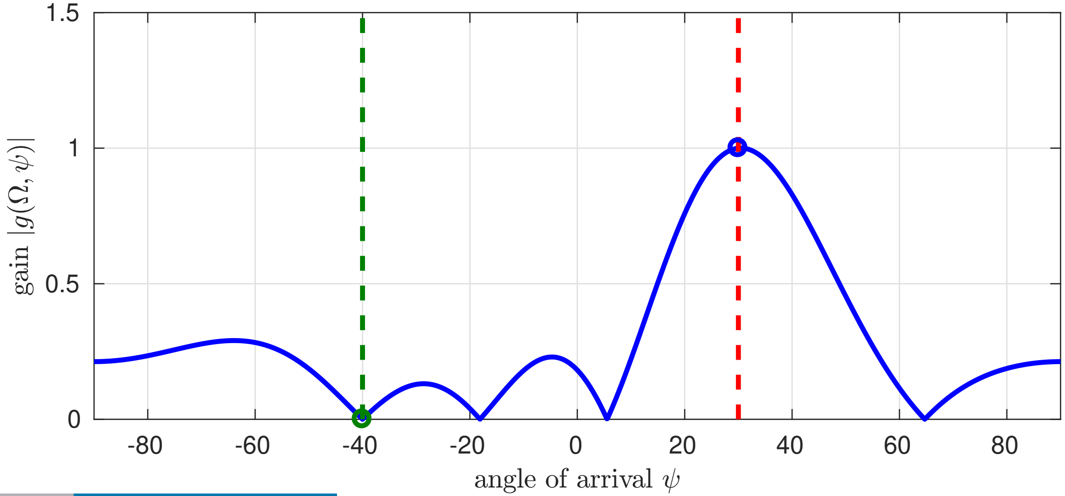

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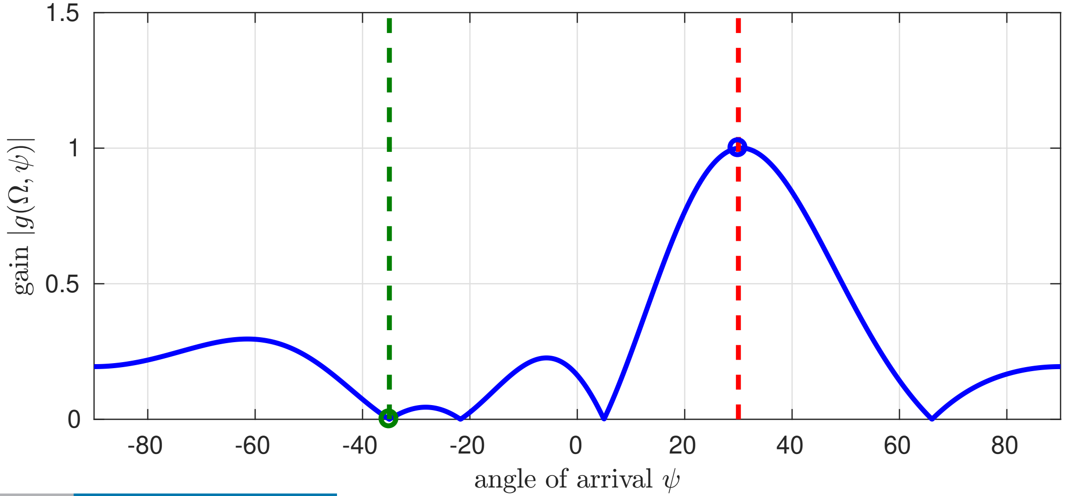

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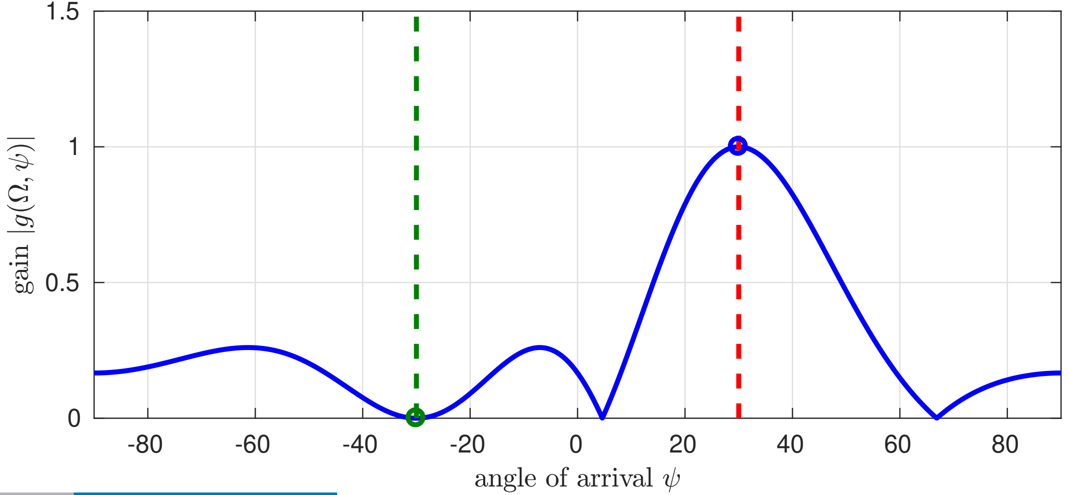

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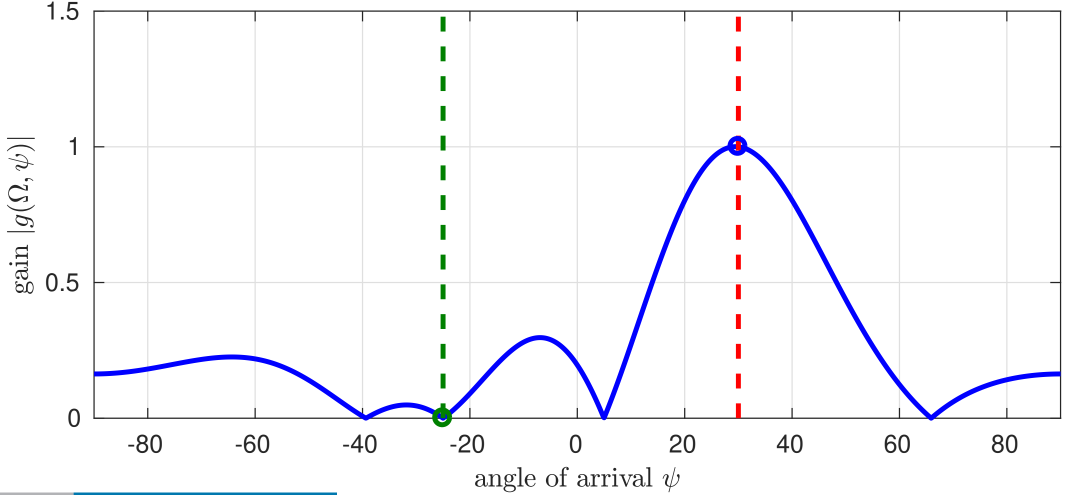

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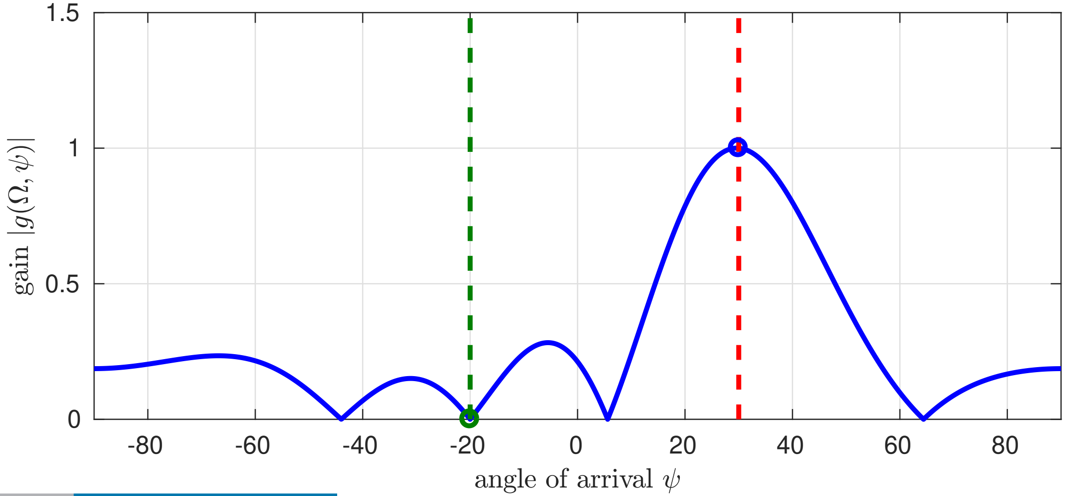

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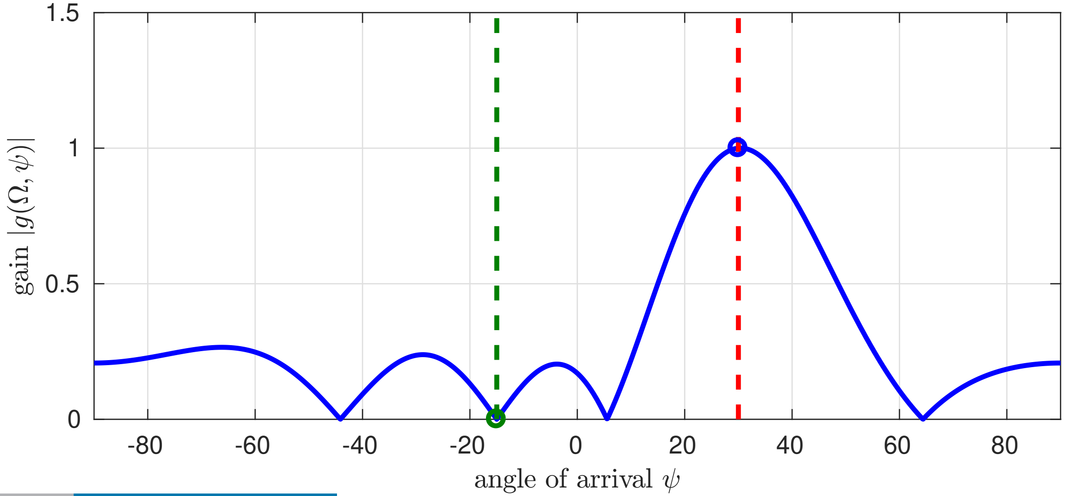

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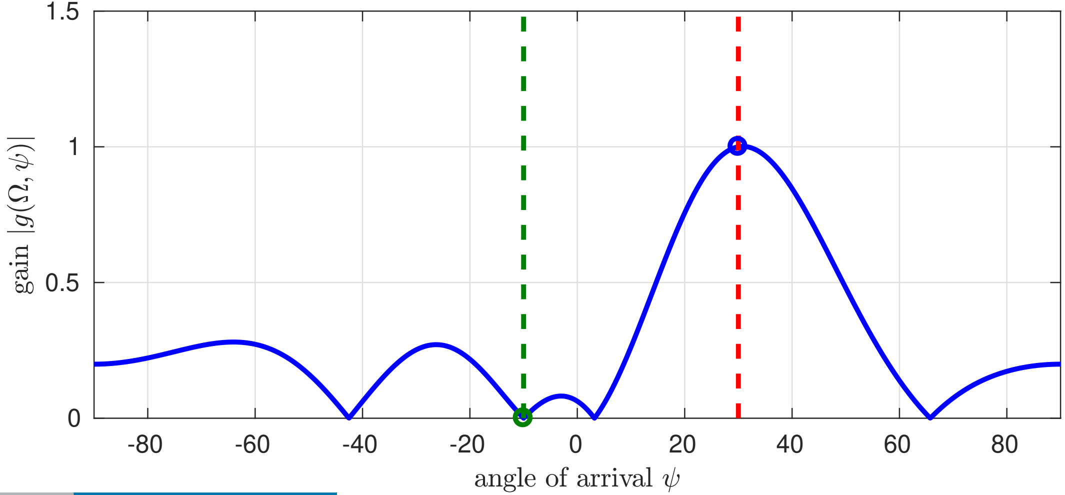

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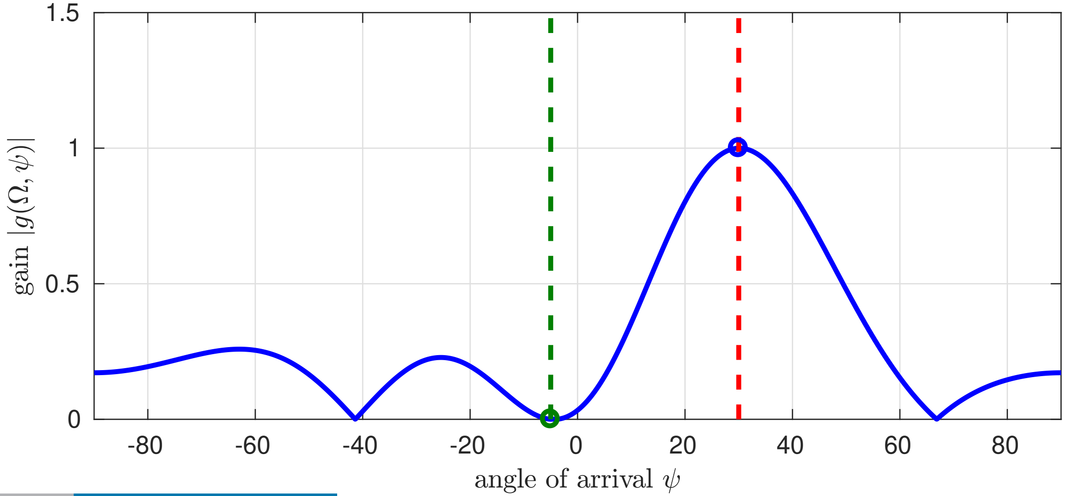

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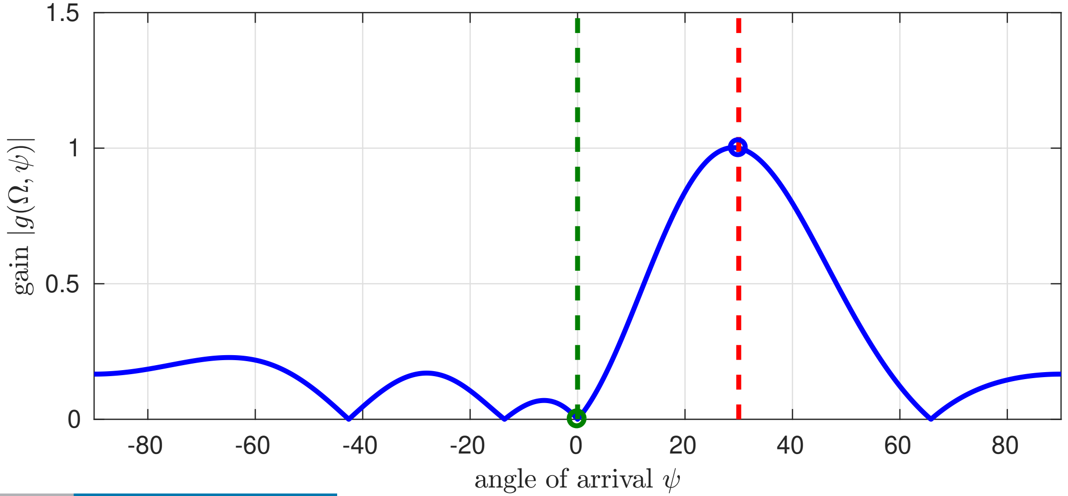

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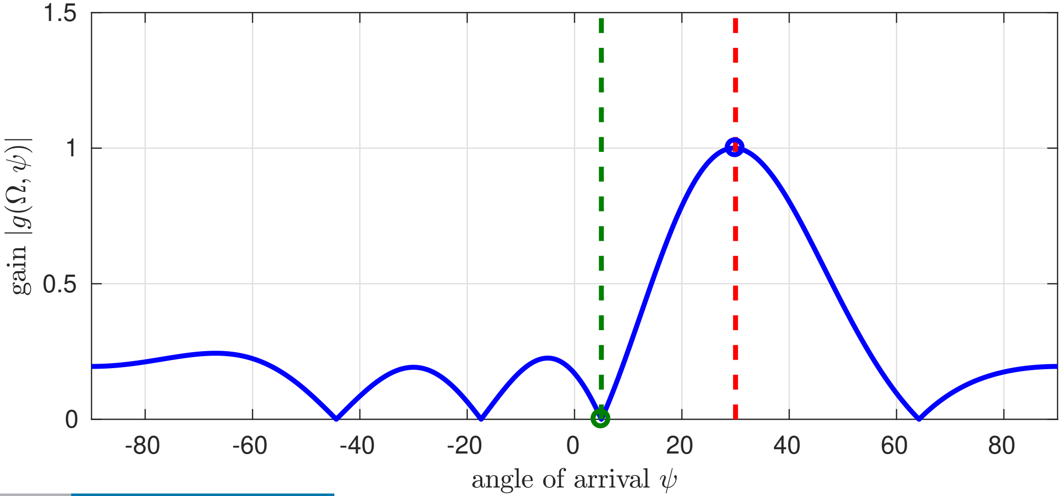

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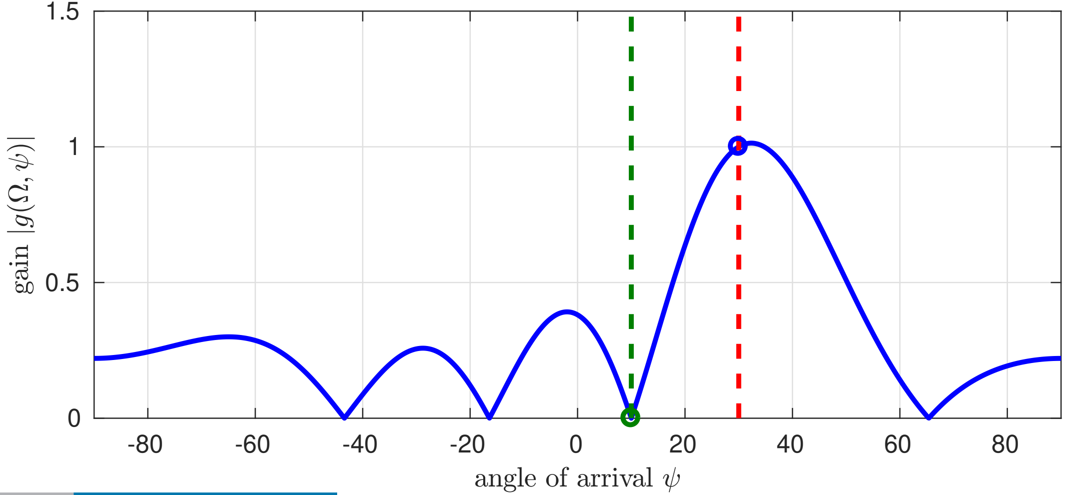

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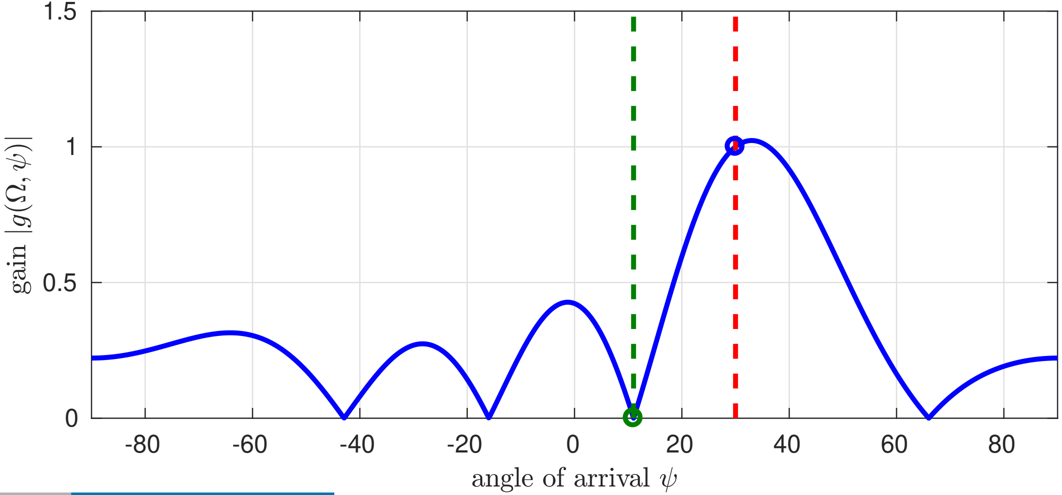

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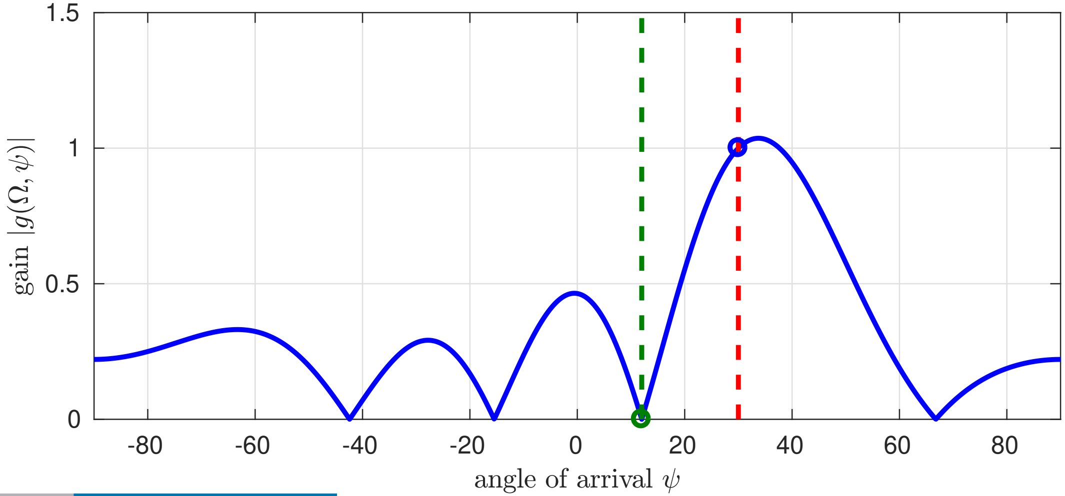

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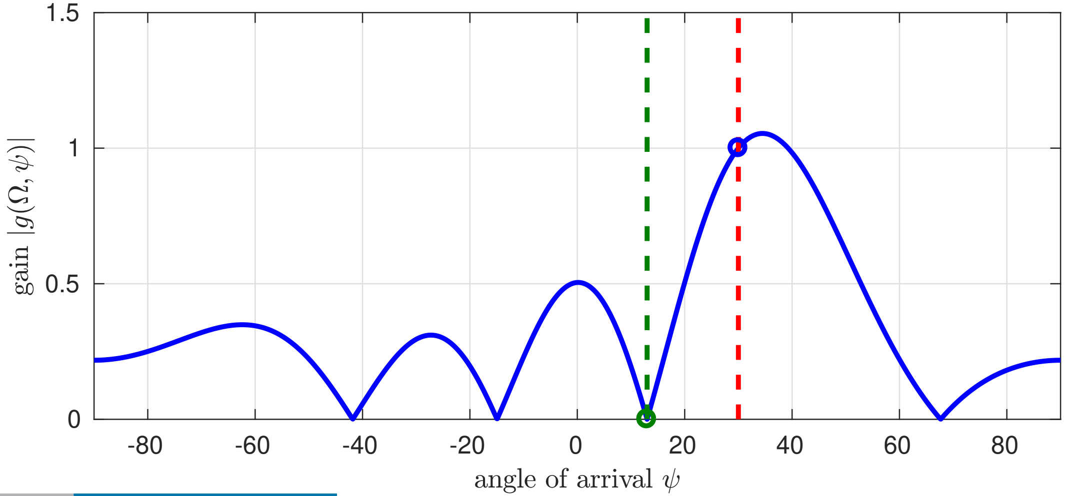

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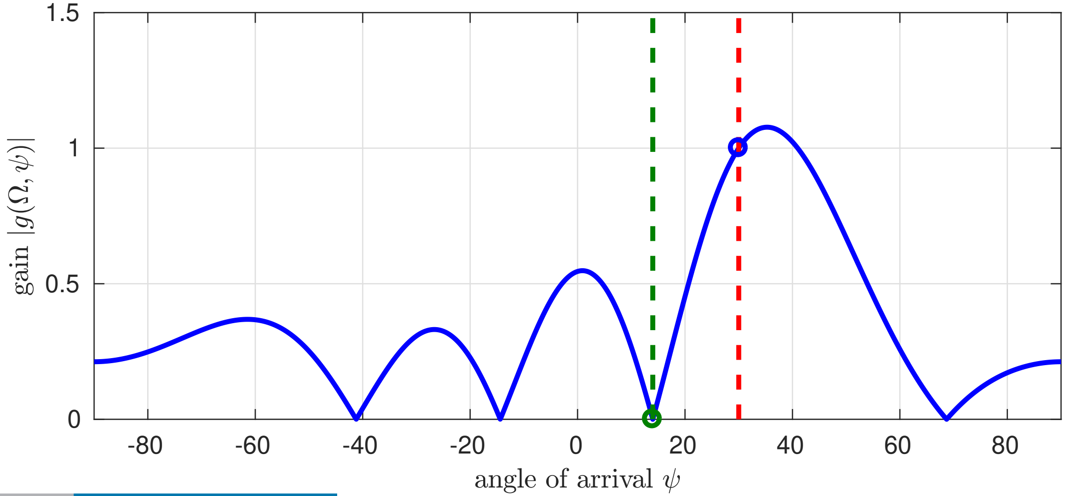

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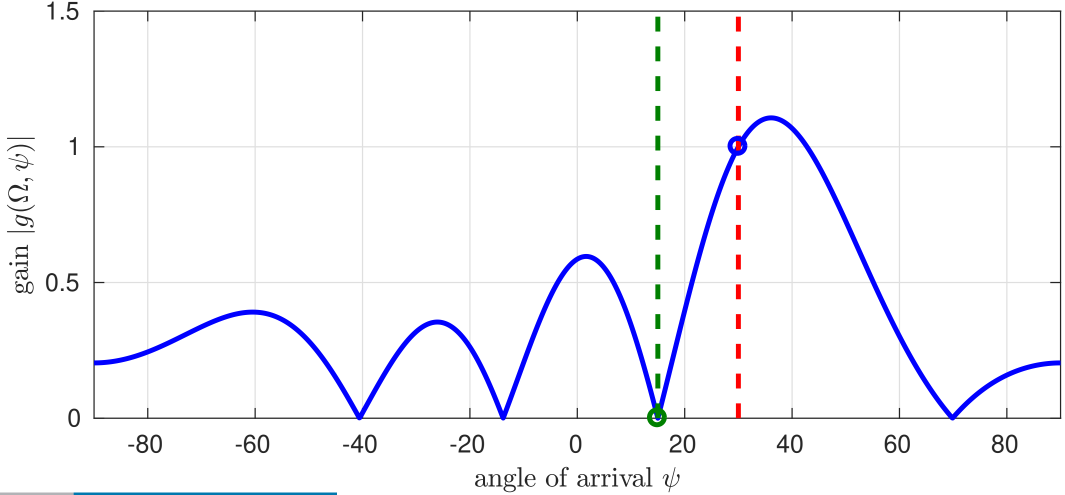

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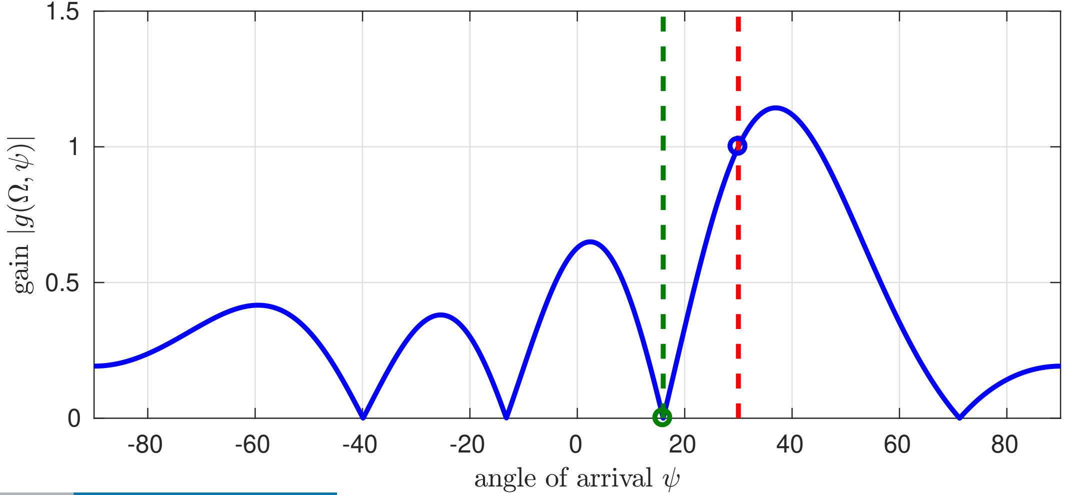

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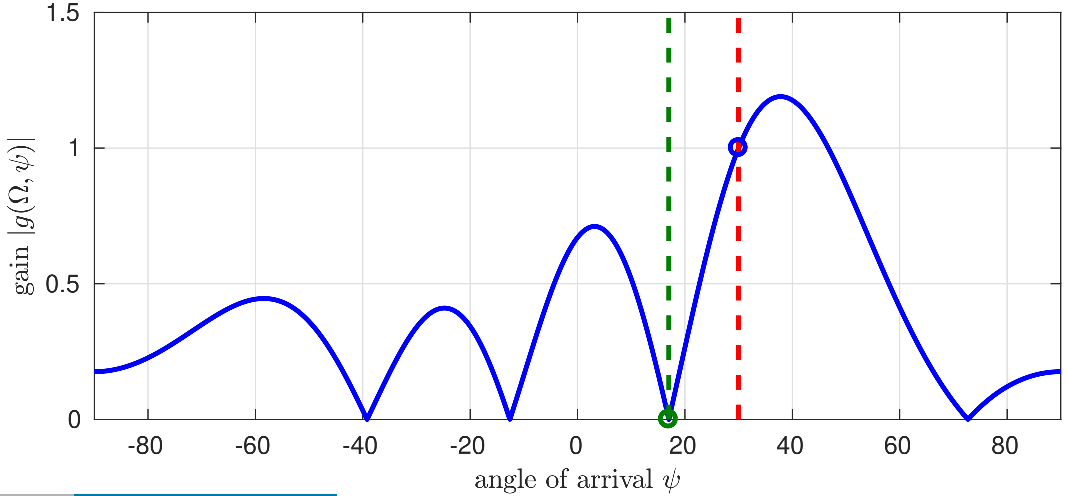

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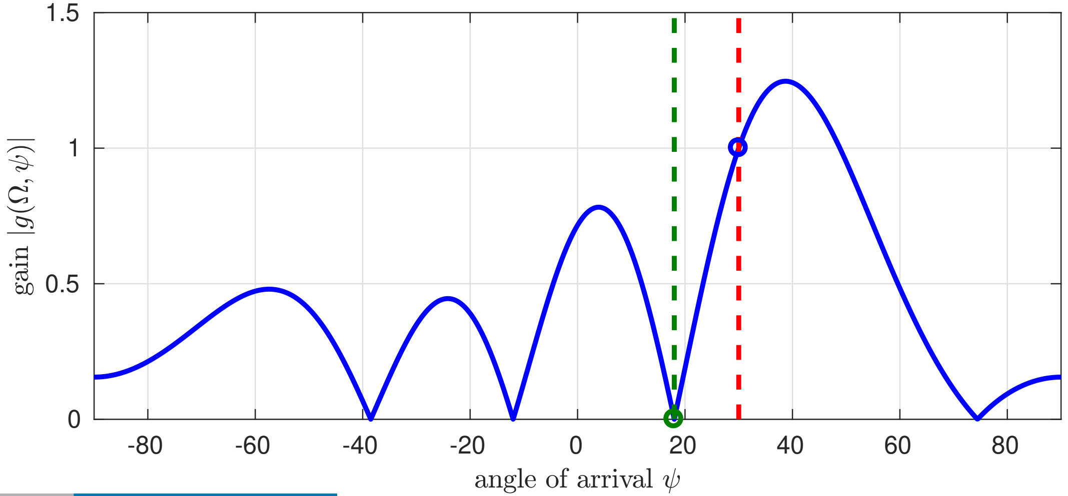

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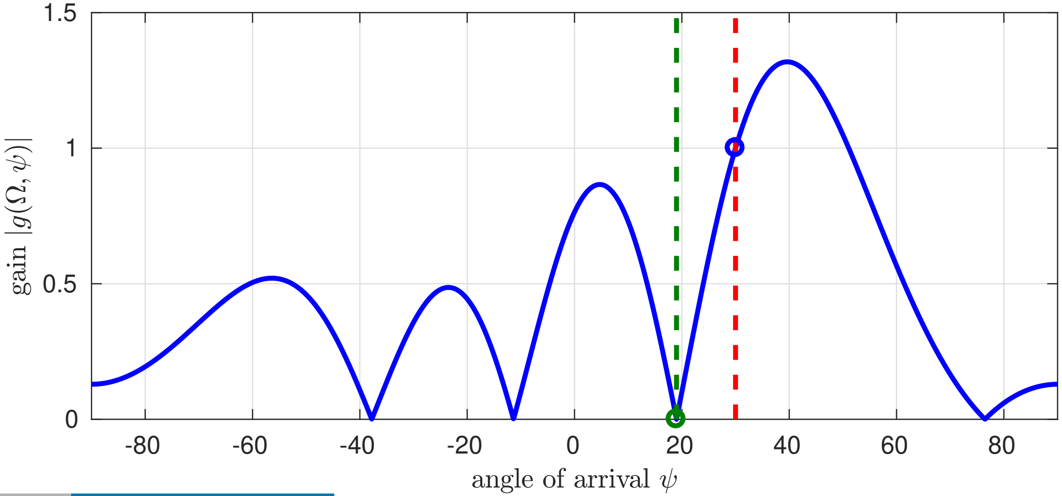

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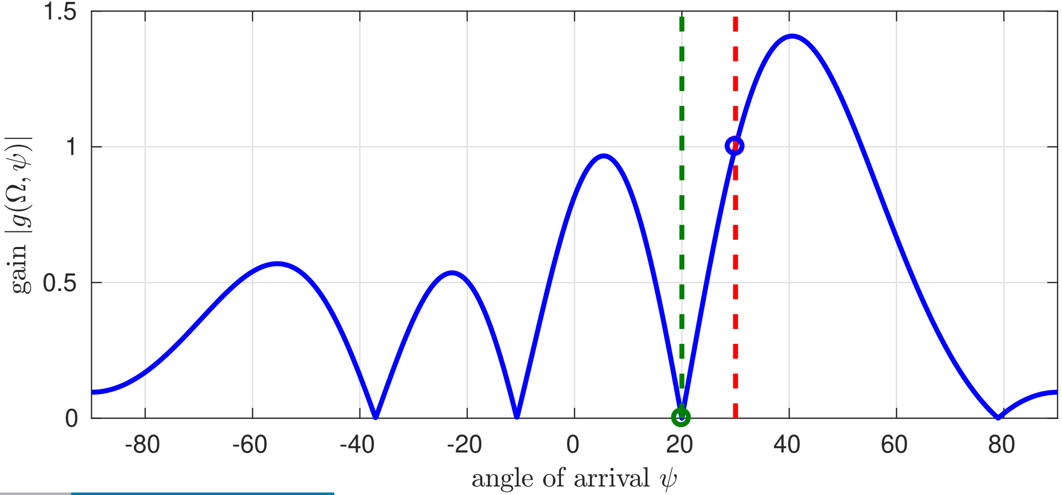

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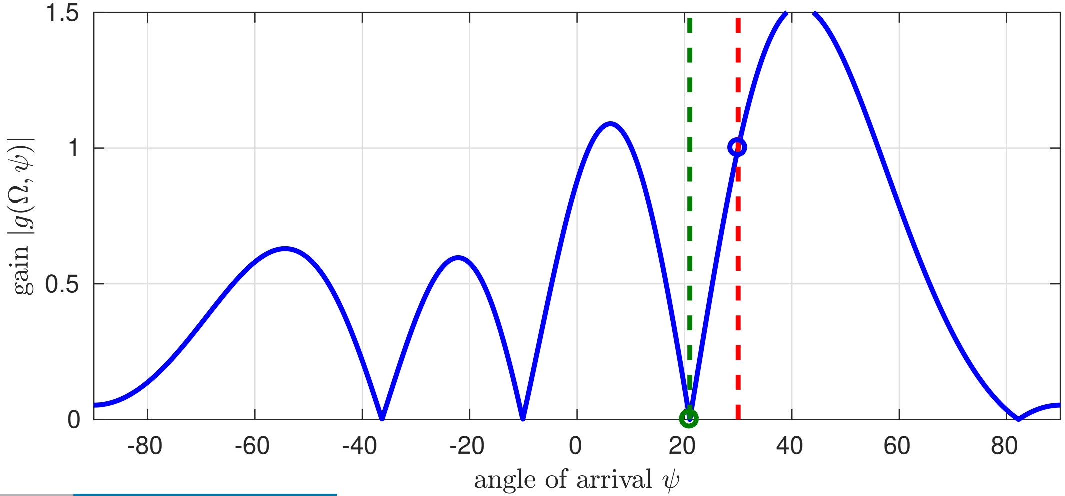
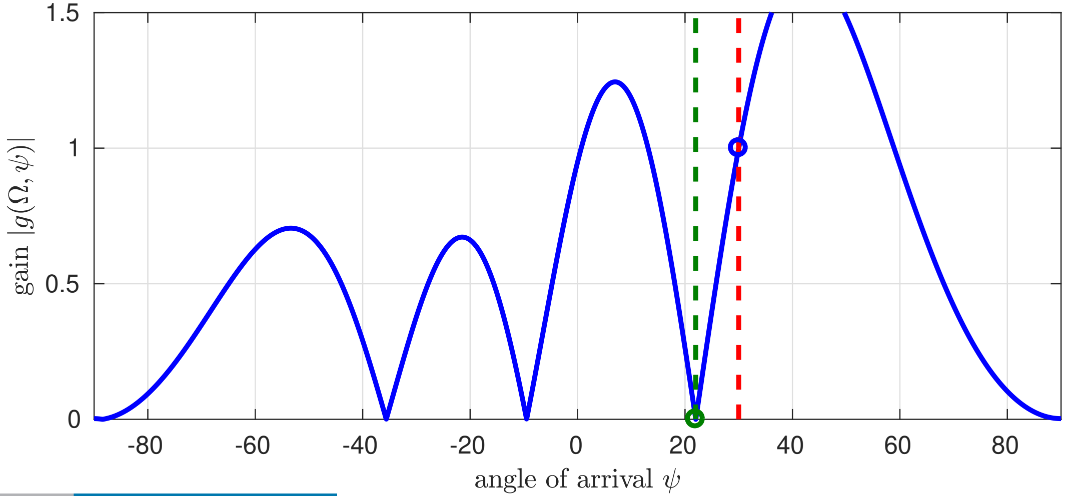


Beamforming Example — Variable Interferer I

- $M = 5$ sensors, source of interest towards $\theta_0 = 30°$, interferer variable:



Beamforming Example — Variable Interferer I

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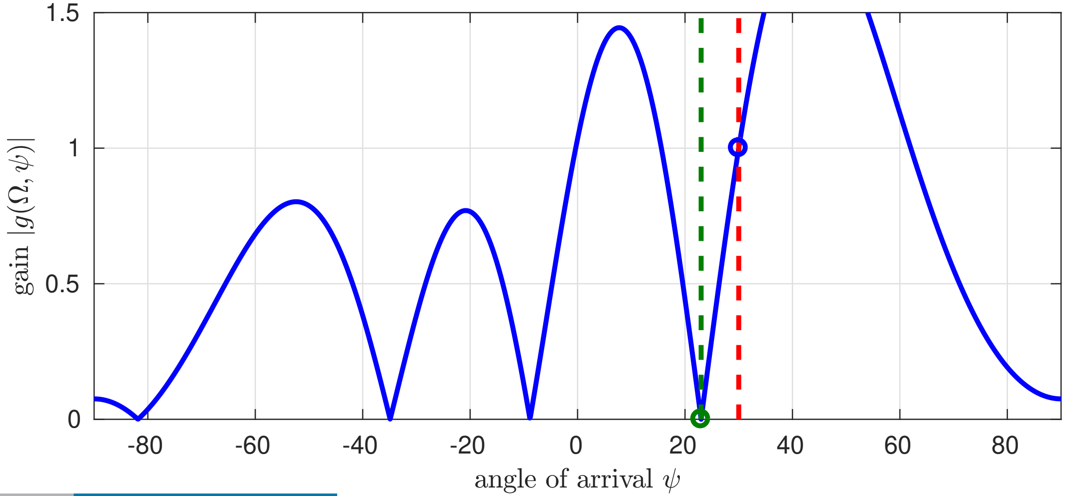

Beamforming Example — Variable Interferer I

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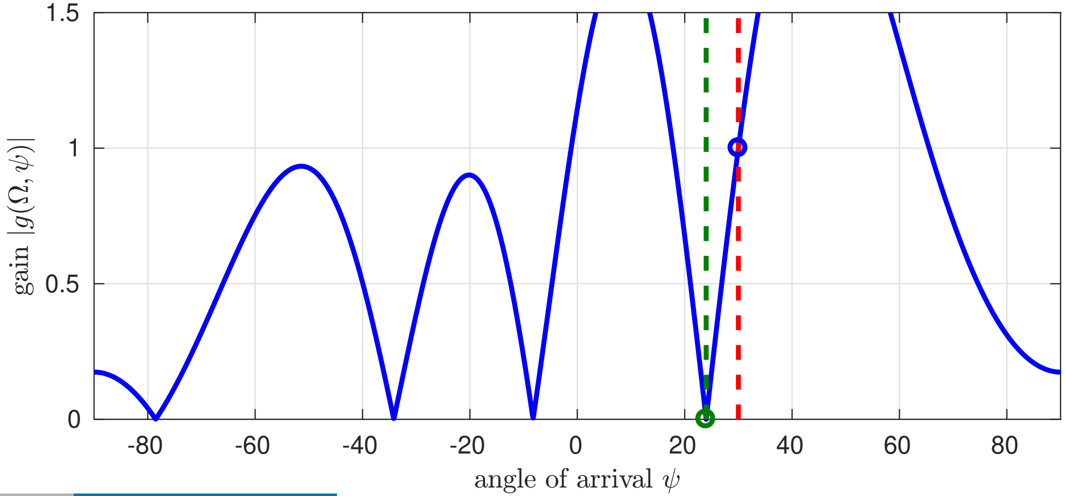

Beamforming Example — Variable Interferer I

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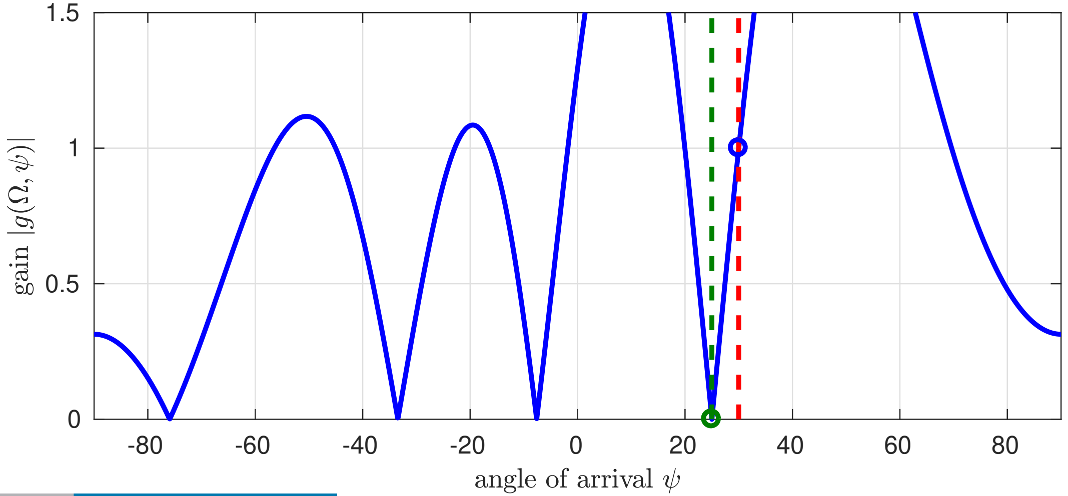
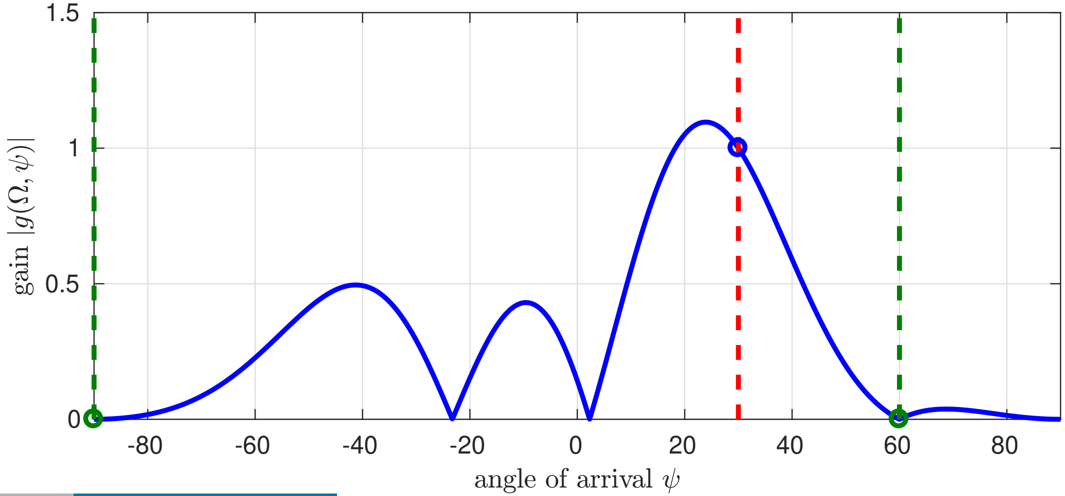


Beamforming Example — Variable Interferer II

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one fixed and one variable interferer:



Beamforming Example — Variable Interferer II

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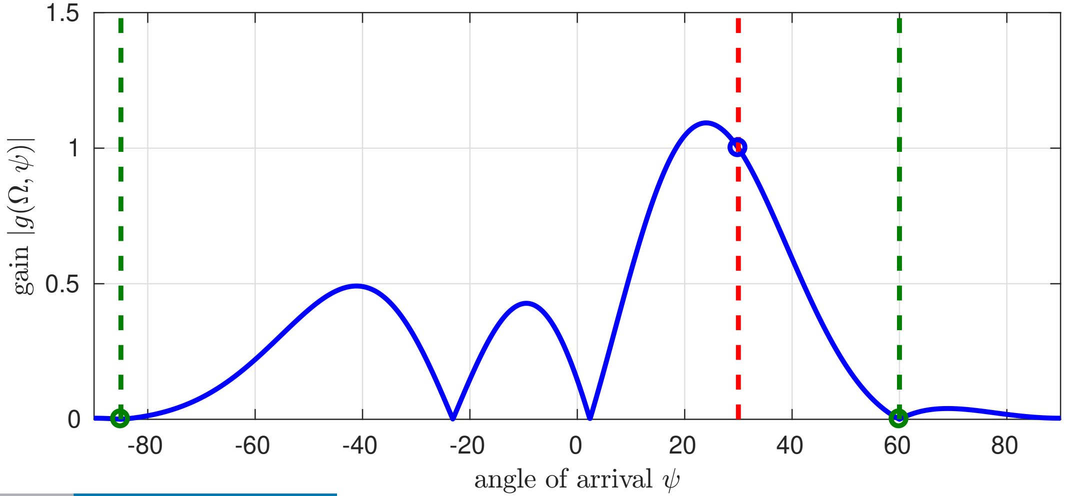

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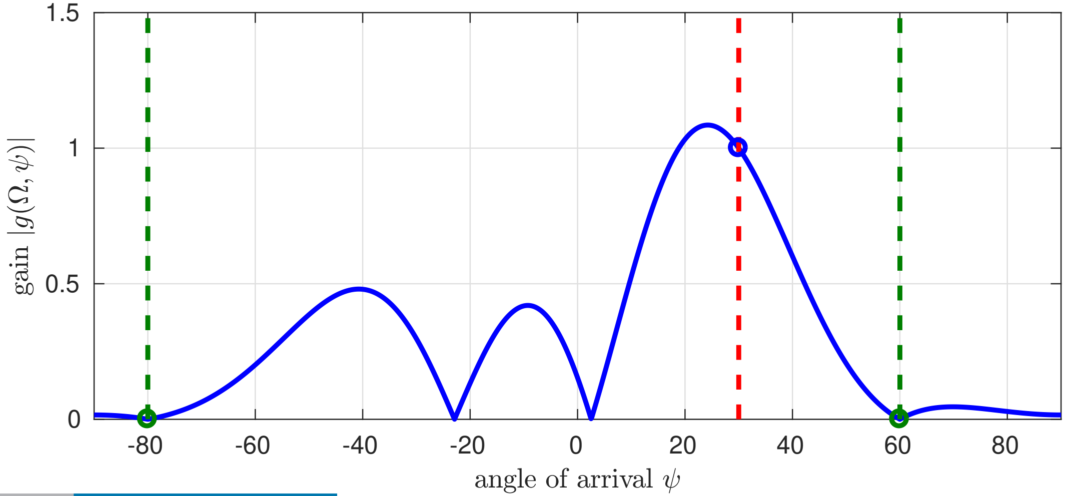

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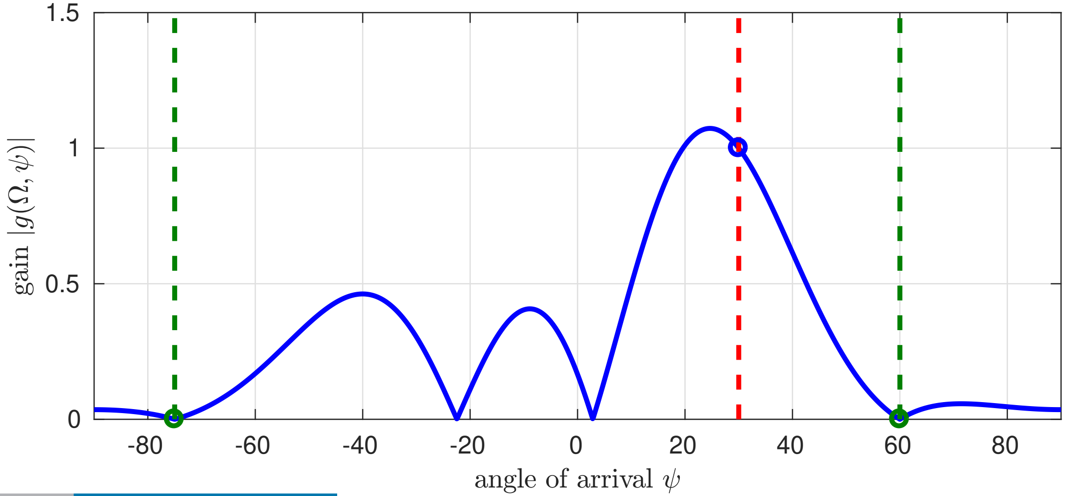

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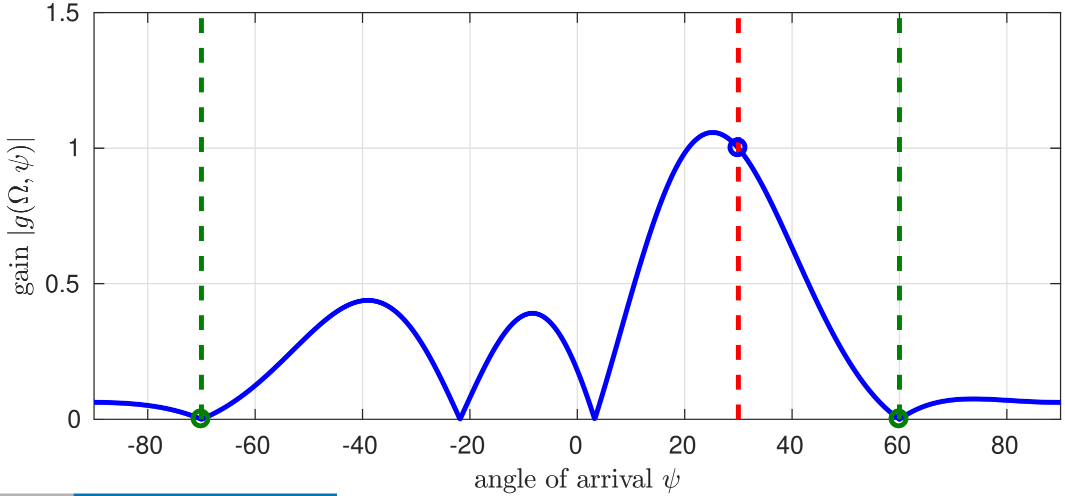

Beamforming Example — Variable Interferer II

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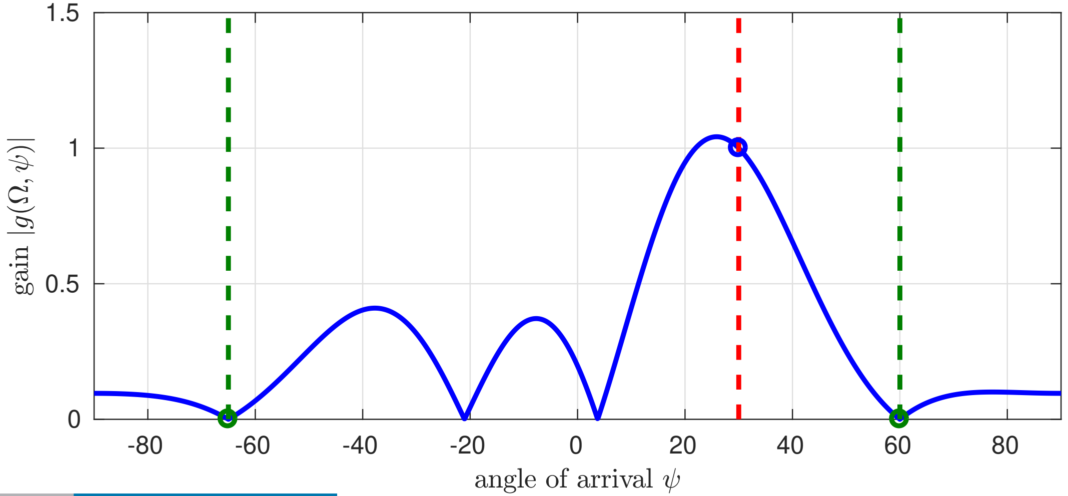

Beamforming Example — Variable Interferer II

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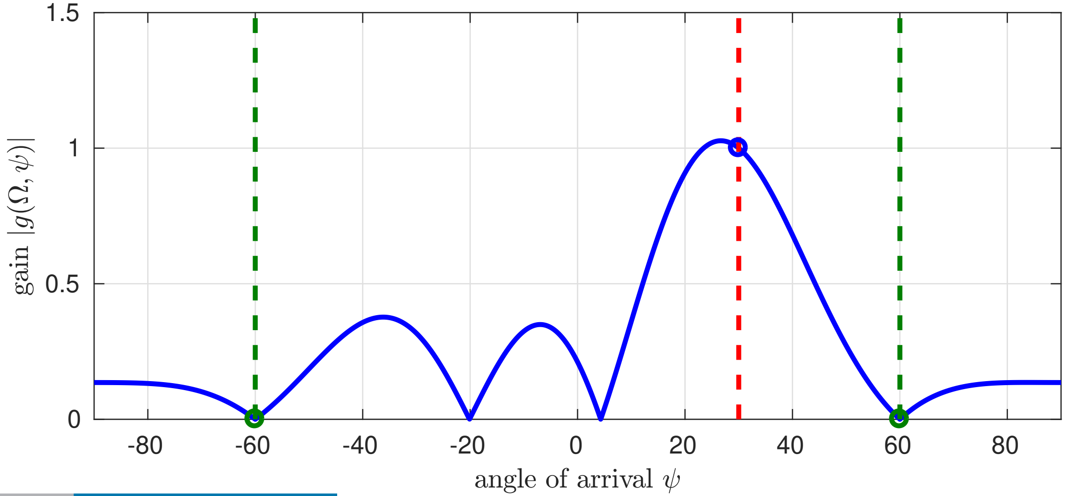

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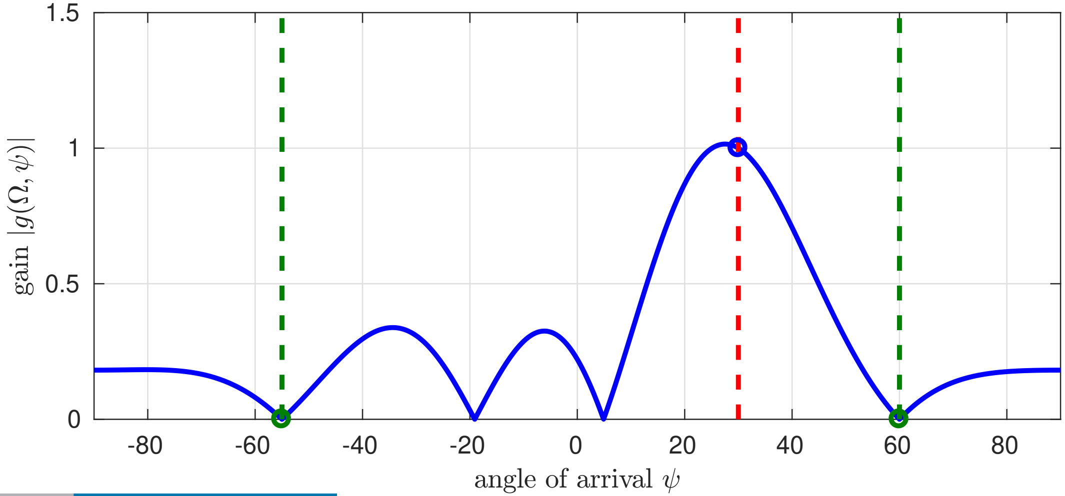

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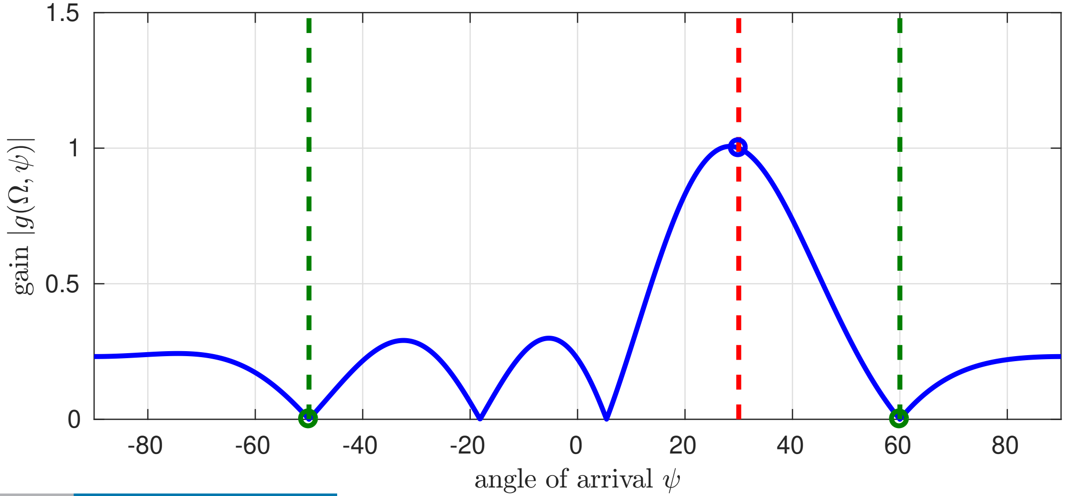

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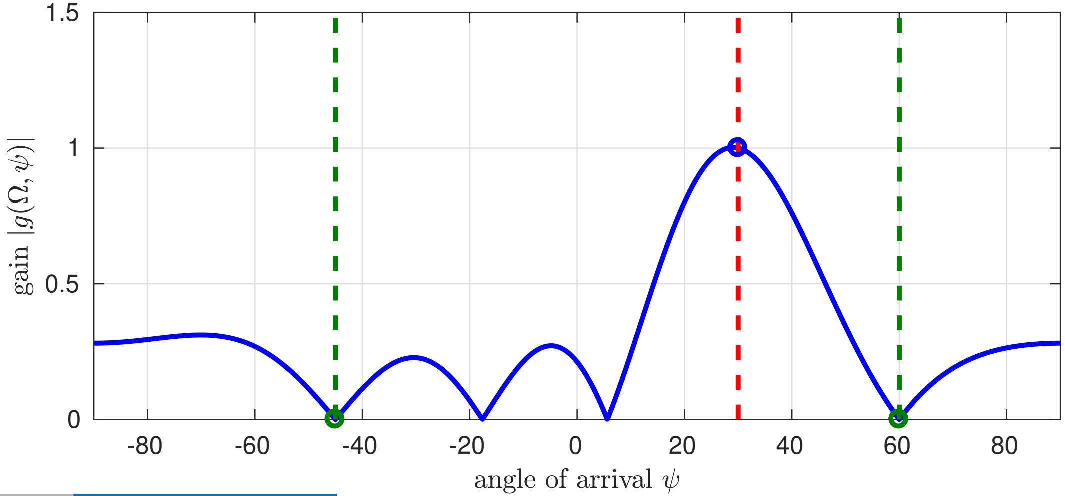

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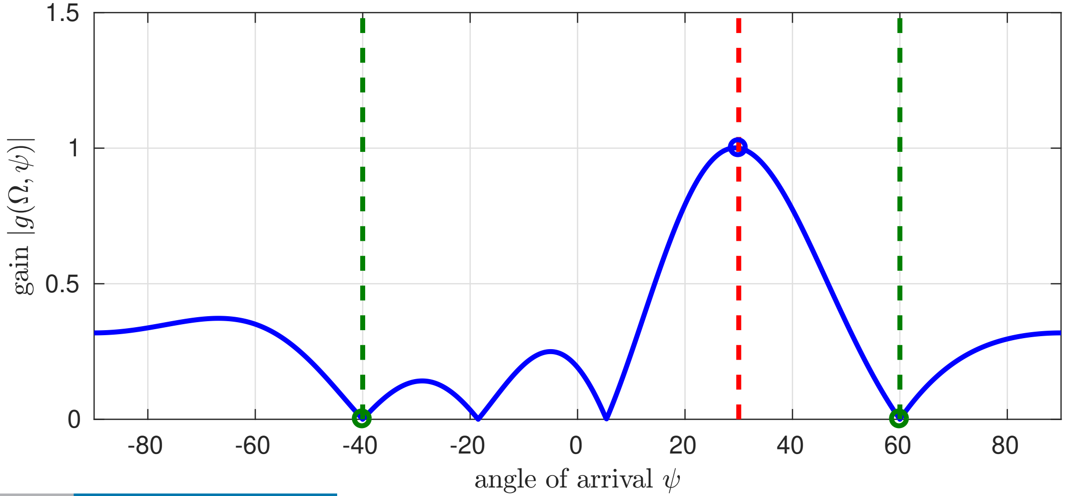

Beamforming Example — Variable Interferer II

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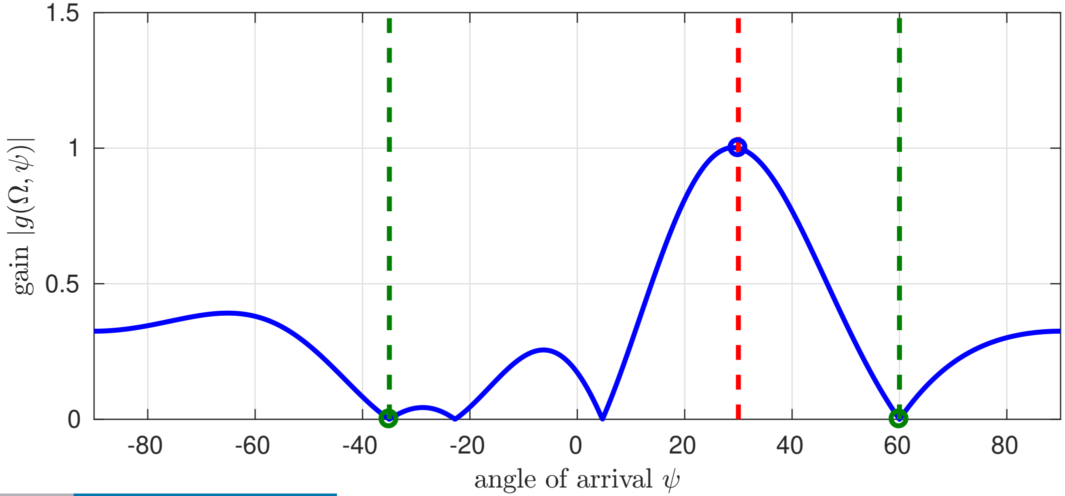

Beamforming Example — Variable Interferer II

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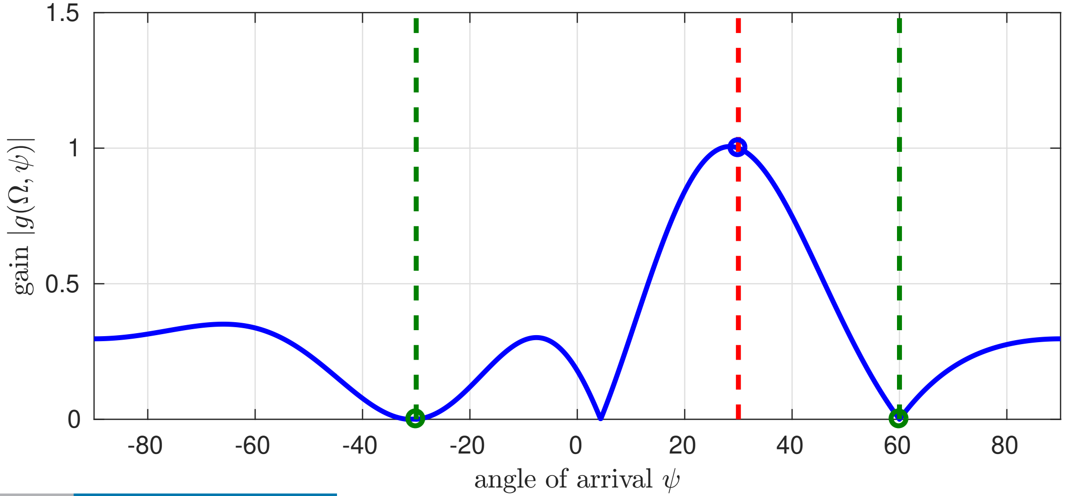

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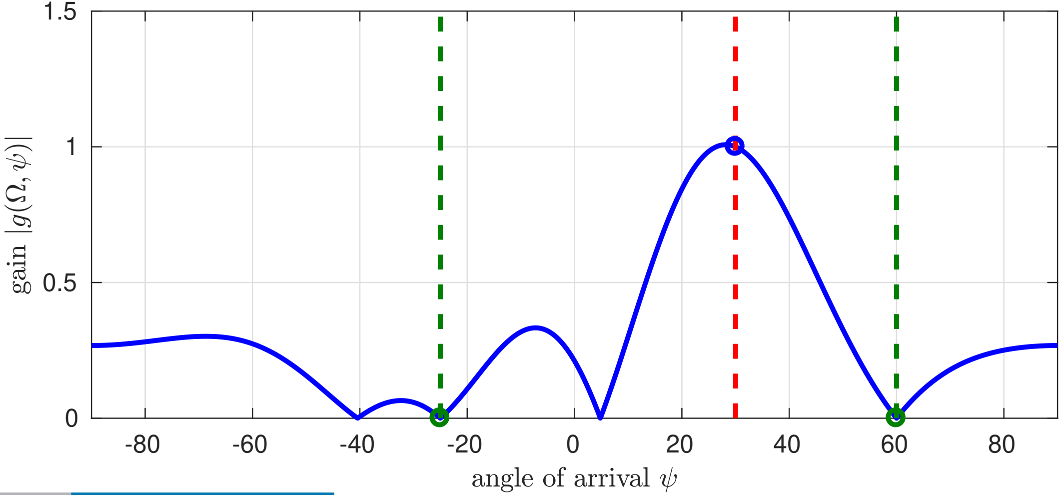

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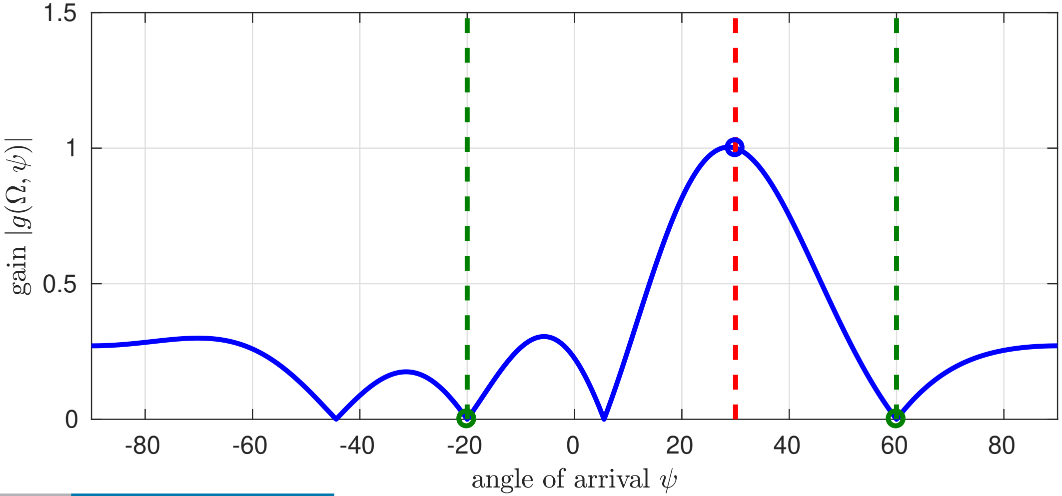

Beamforming Example — Variable Interferer II

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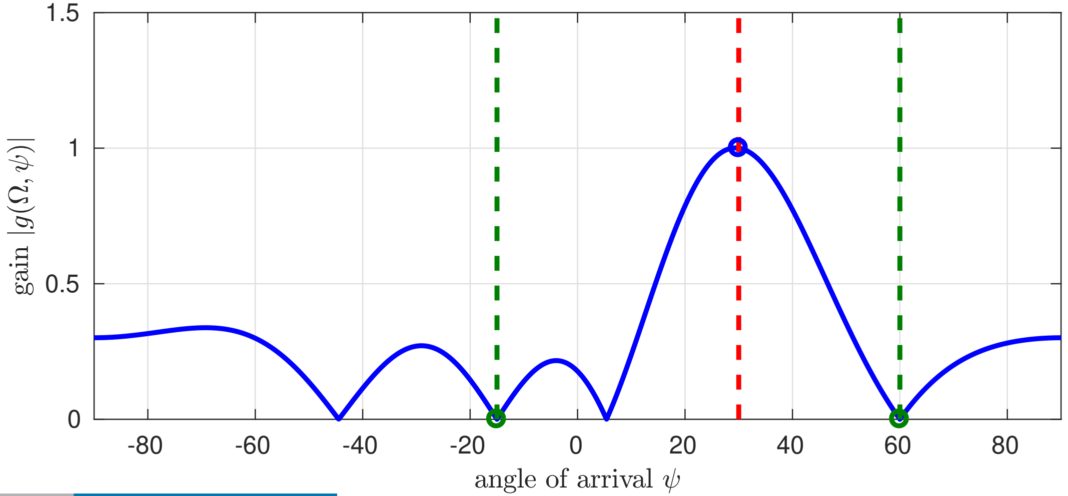

Beamforming Example — Variable Interferer II

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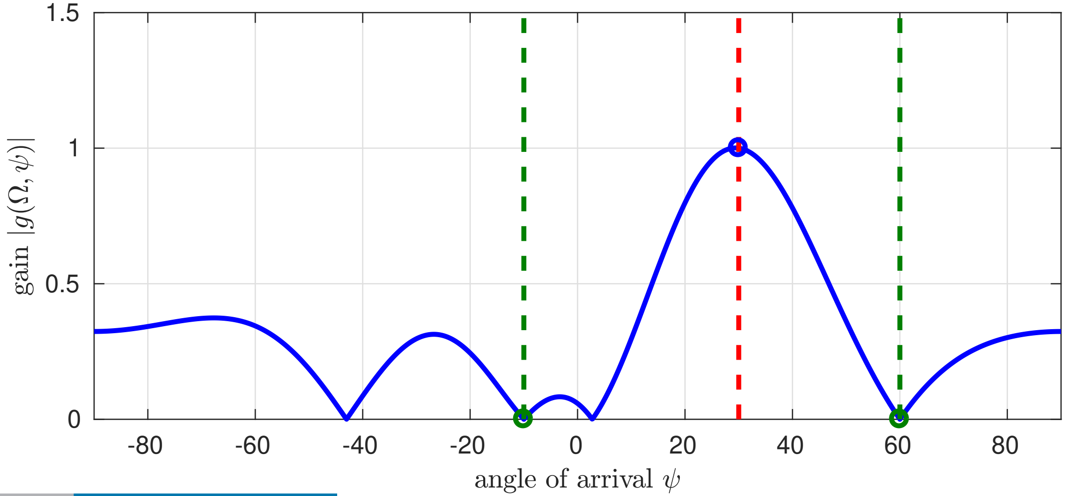

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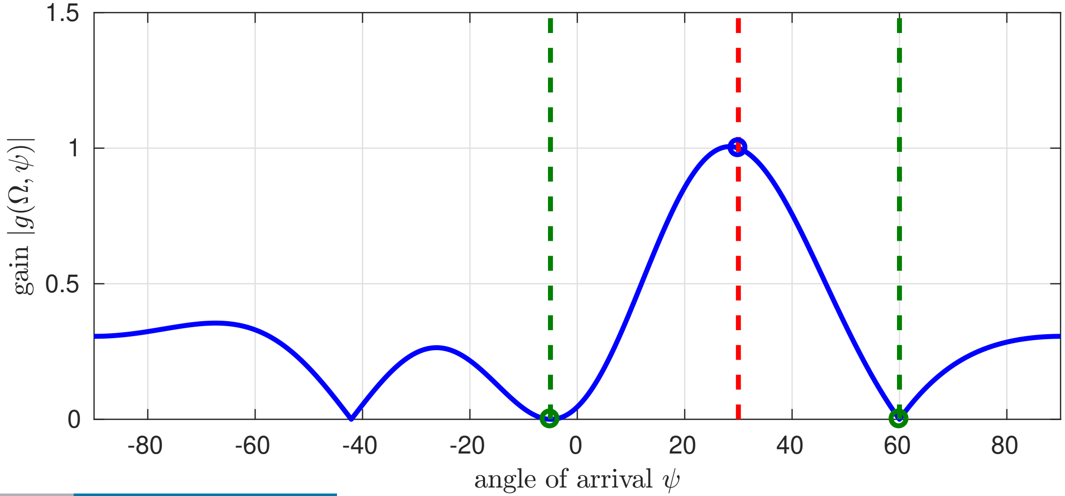

Beamforming Example — Variable Interferer II

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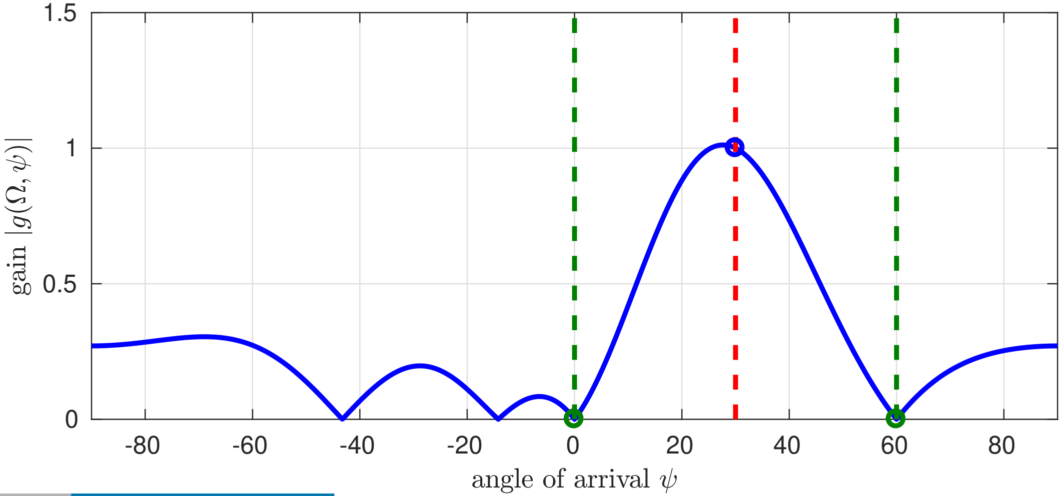

Beamforming Example — Variable Interferer II

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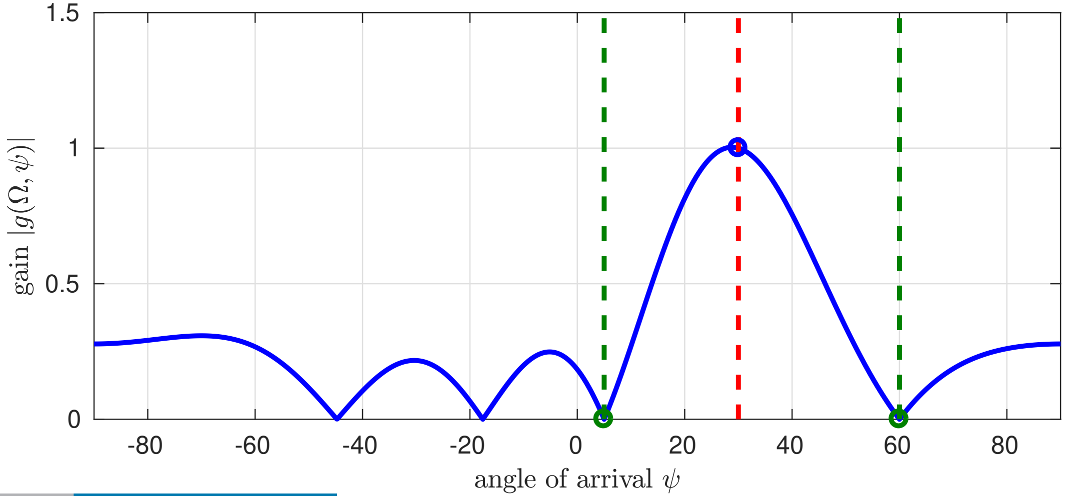
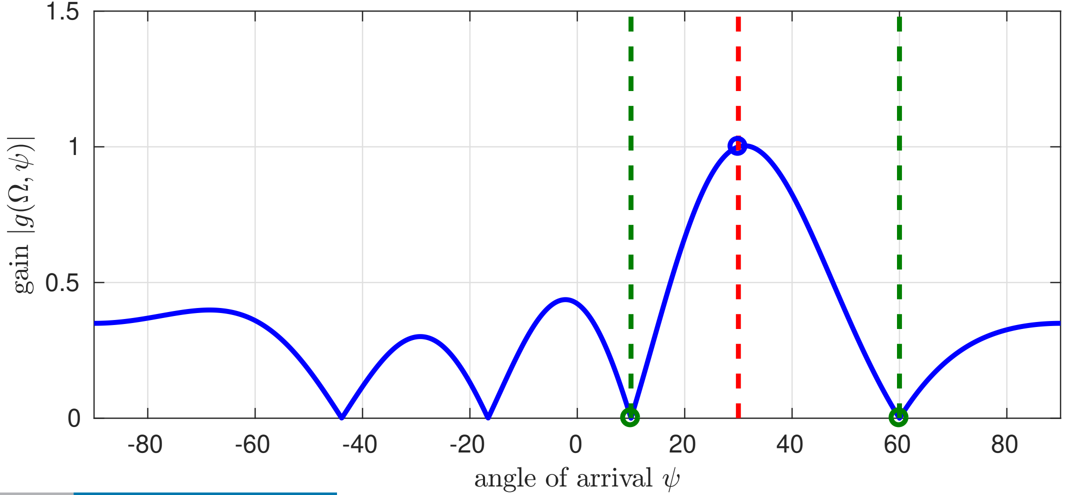


Beamforming Example — Variable Interferer II

- $M = 5$ sensors, SOI $\theta_0 = 30^\circ$, one fixed and one variable interferer:



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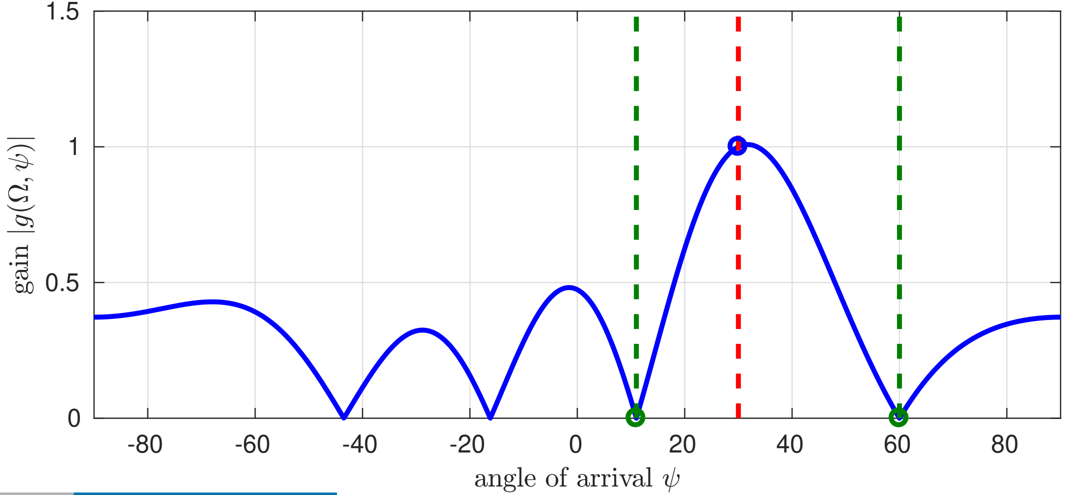

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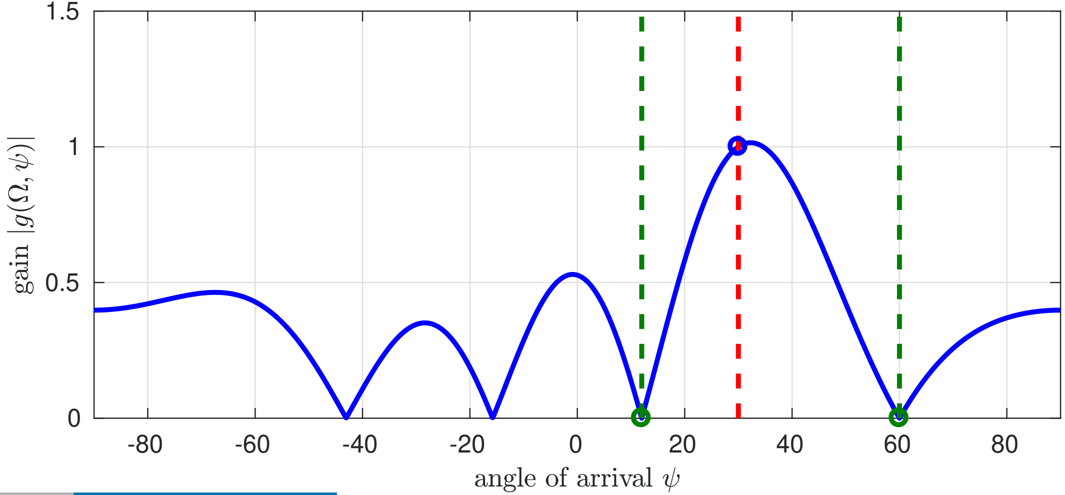

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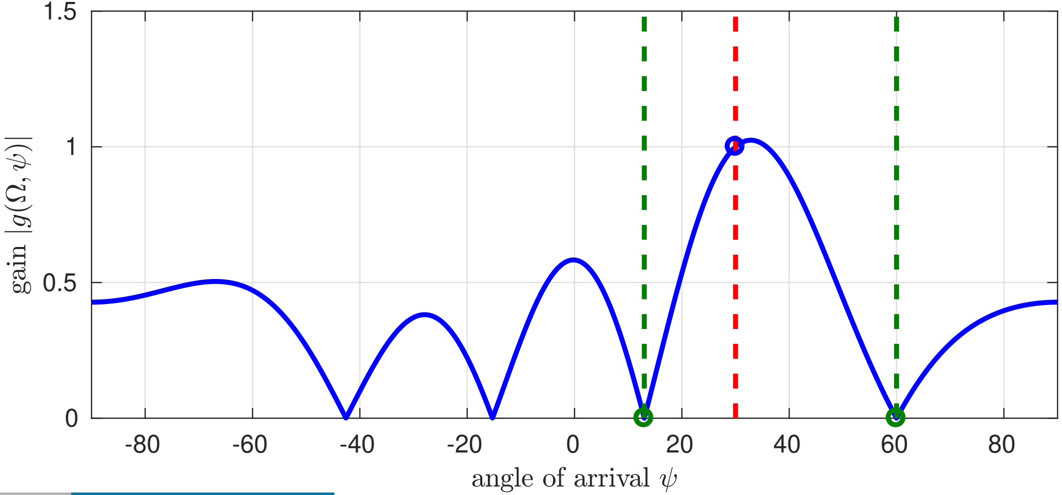

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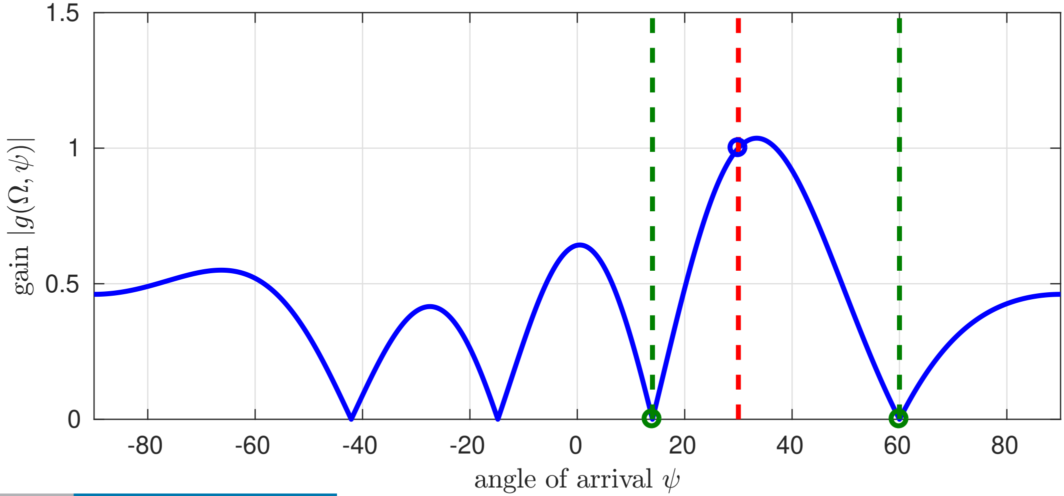

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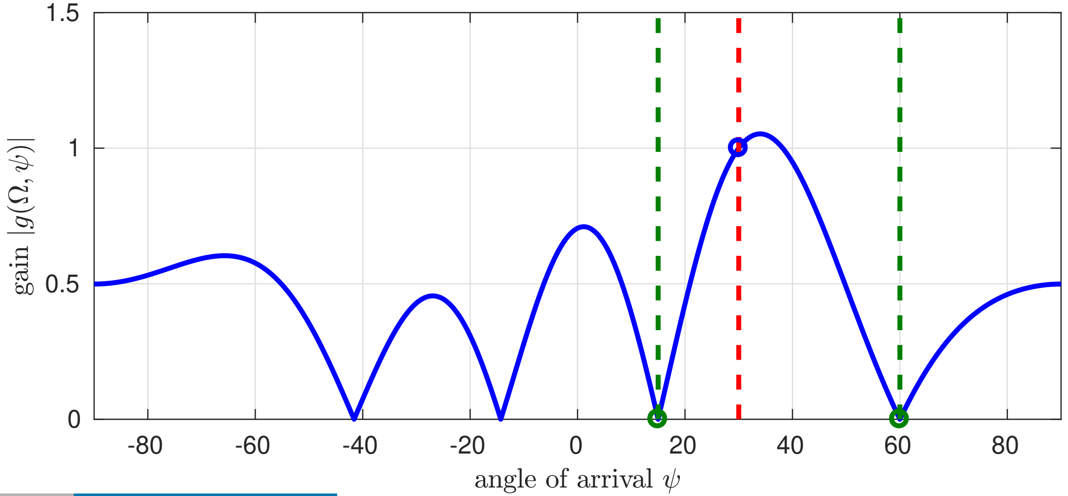

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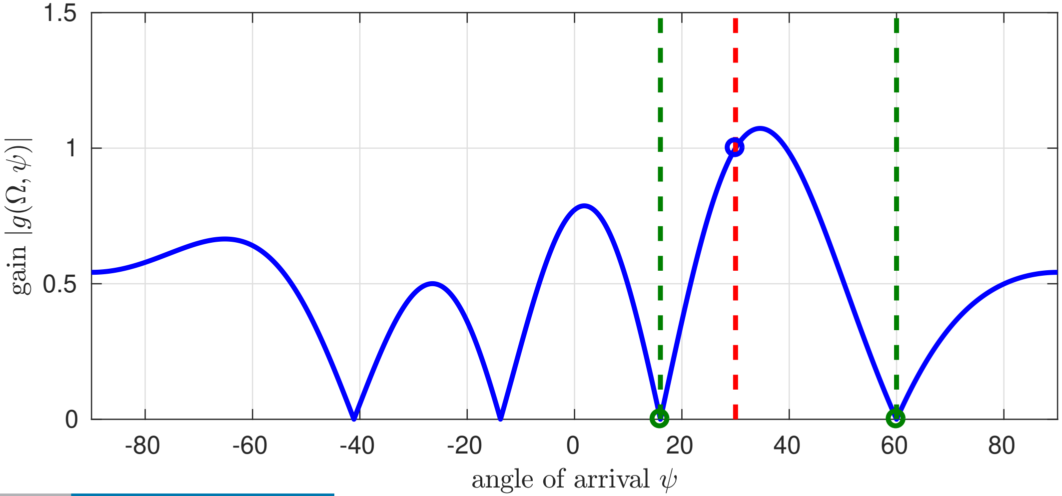

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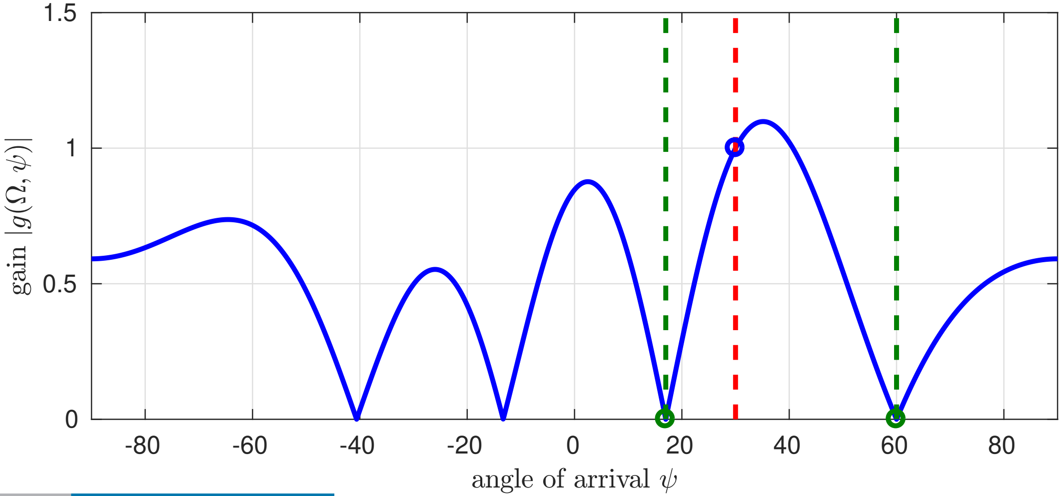

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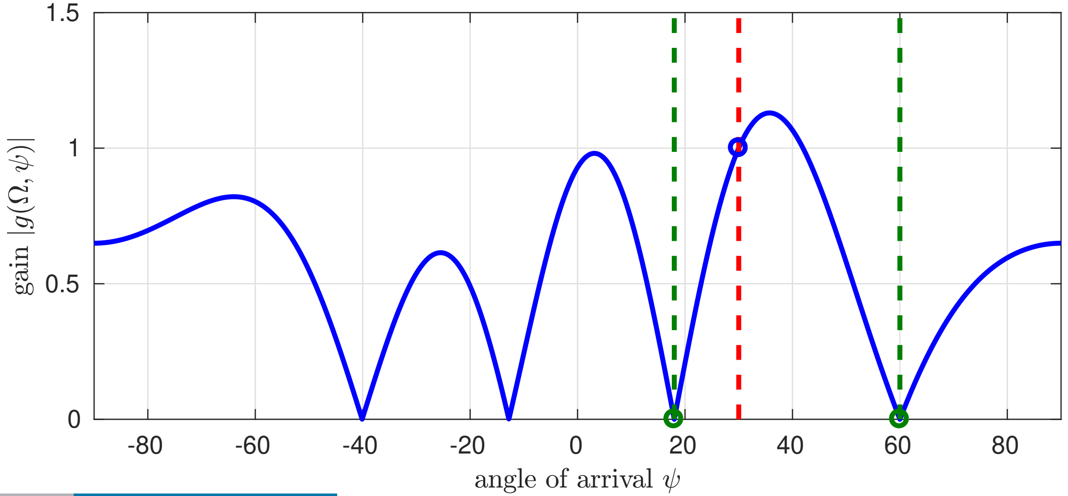

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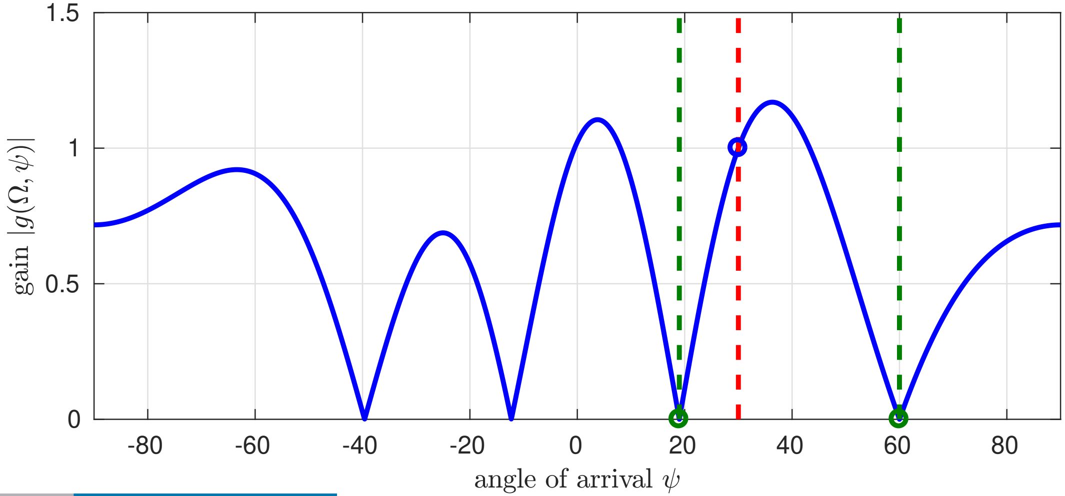

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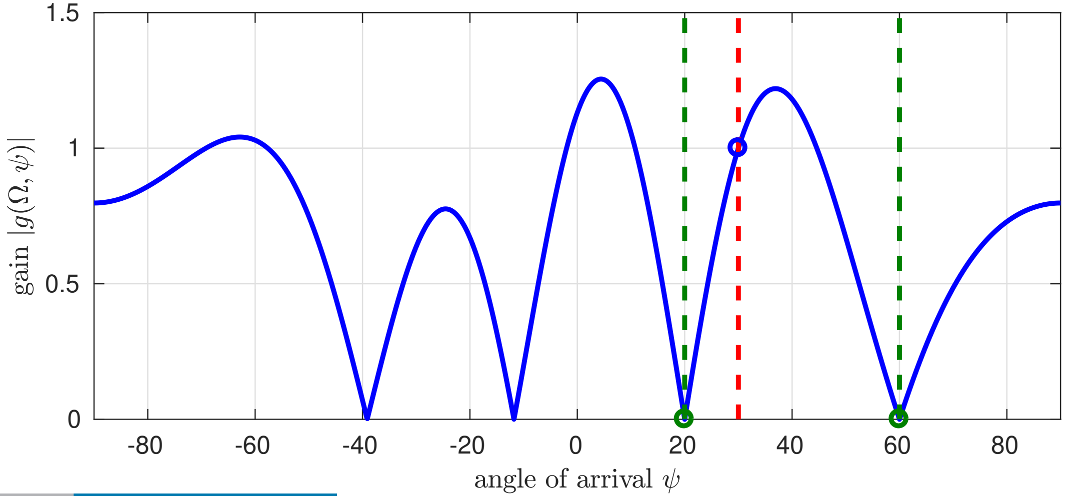

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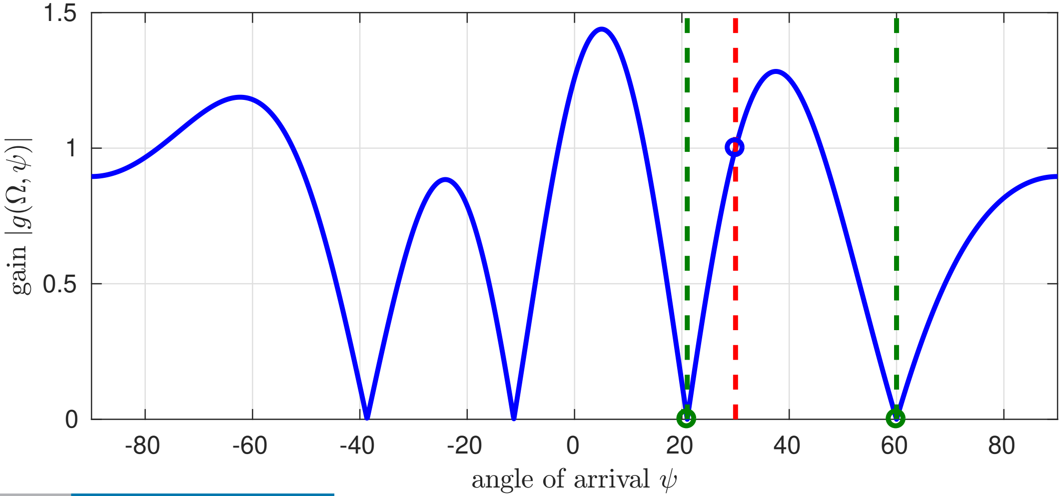

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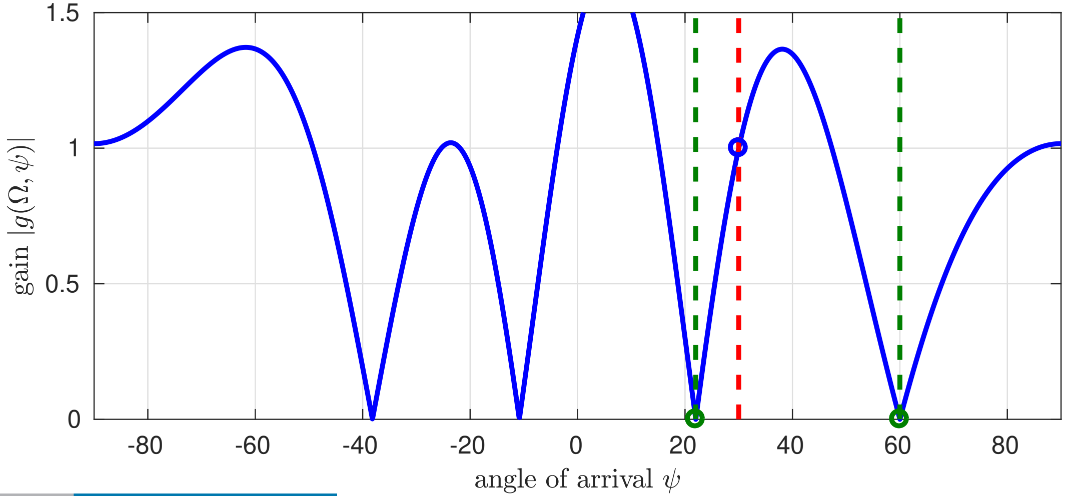

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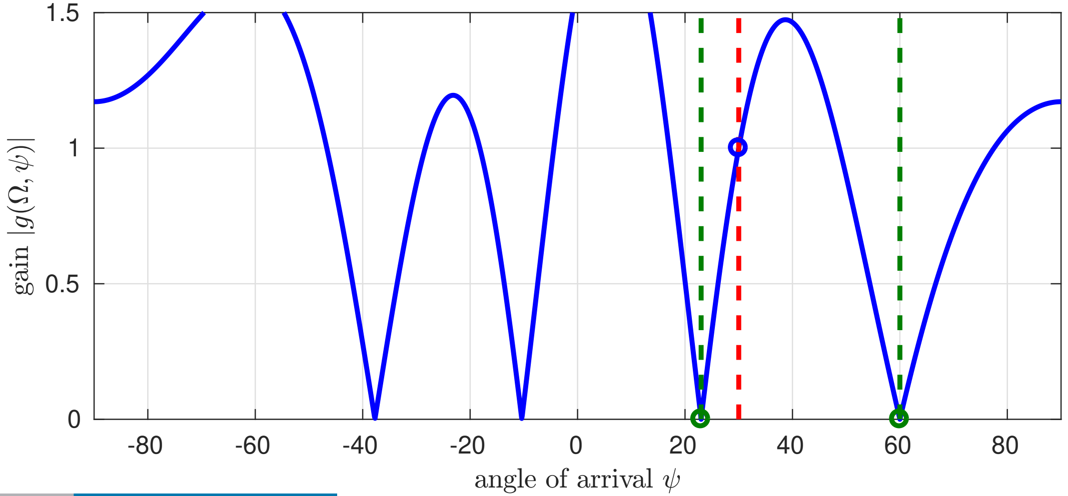

Beamforming Example — Variable Interferer II

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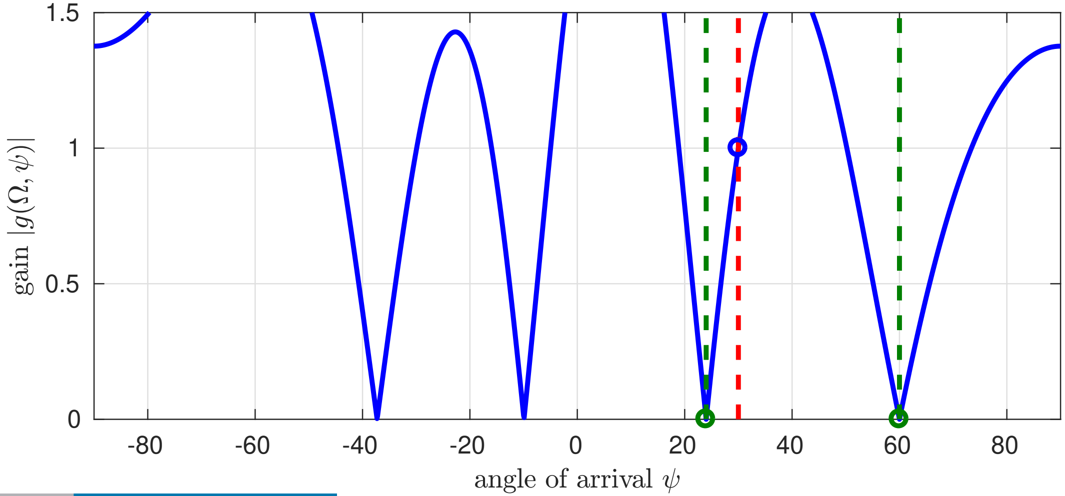

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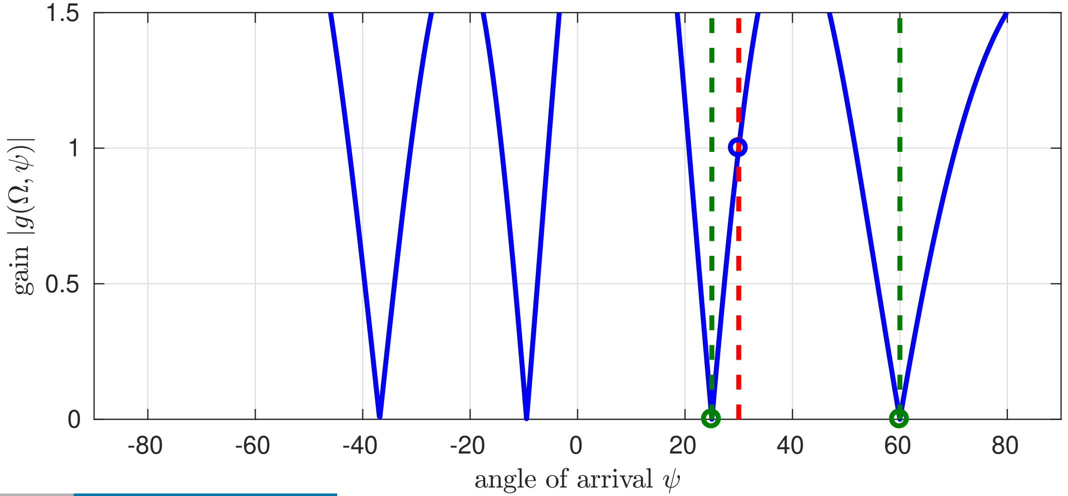
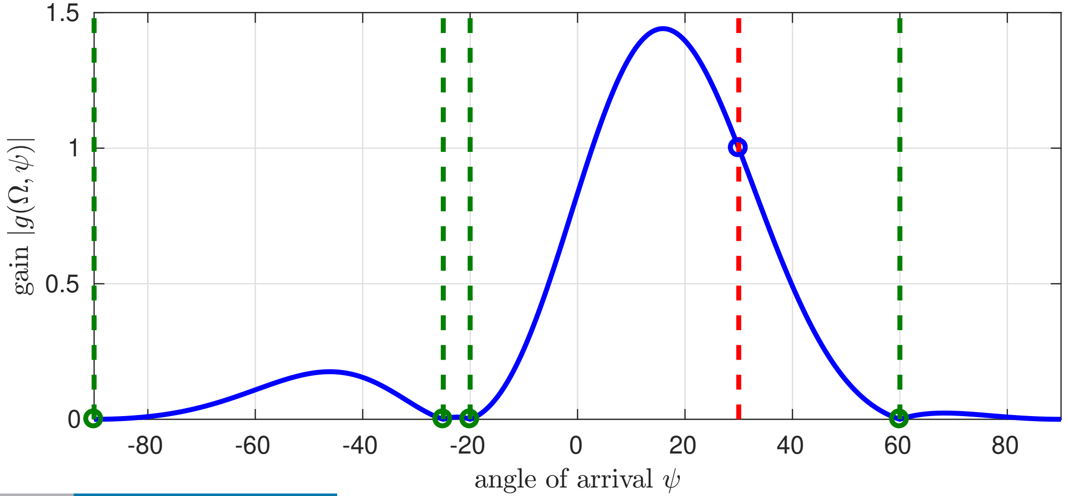


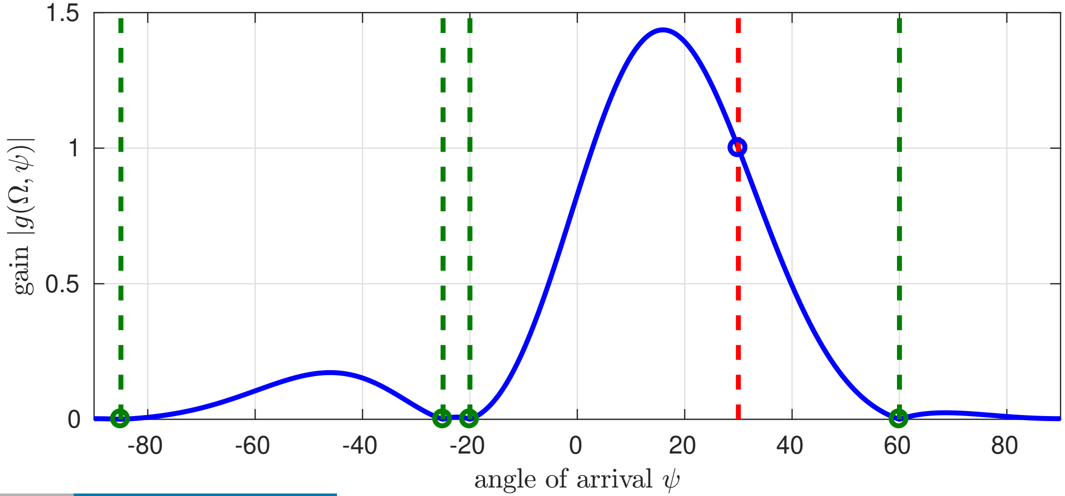
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



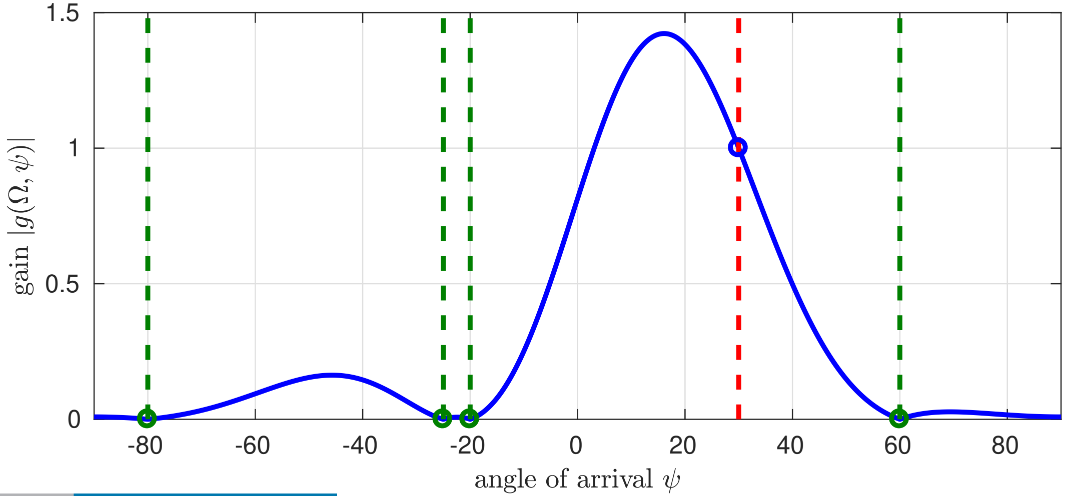
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



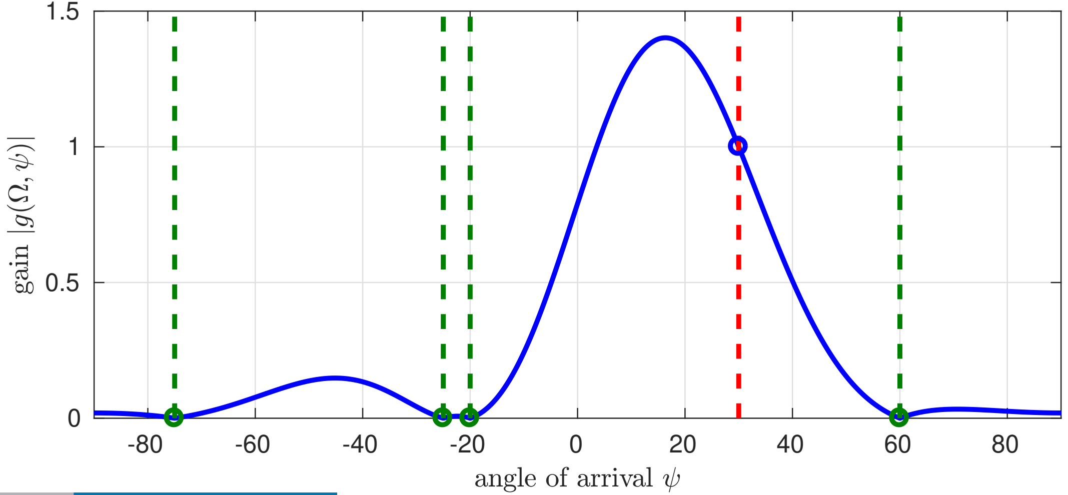
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



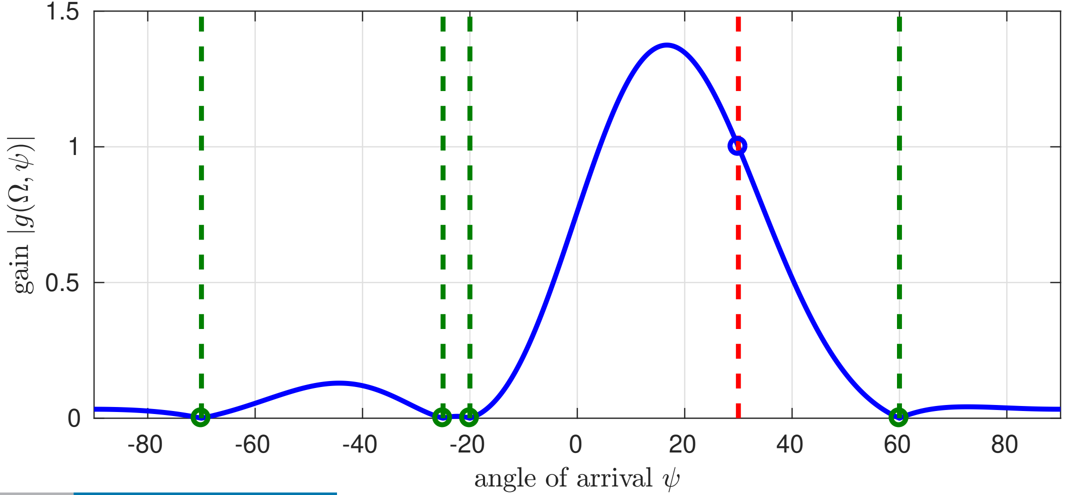
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



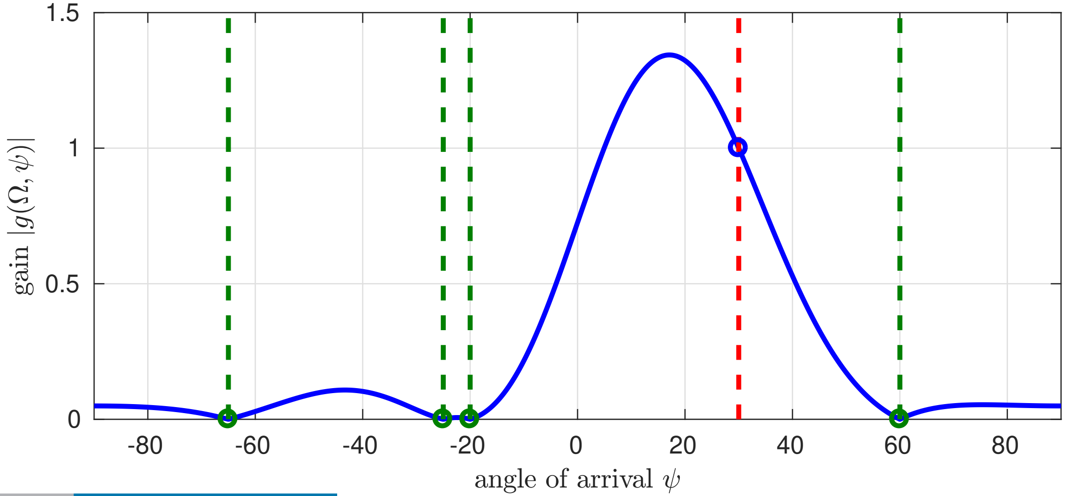
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



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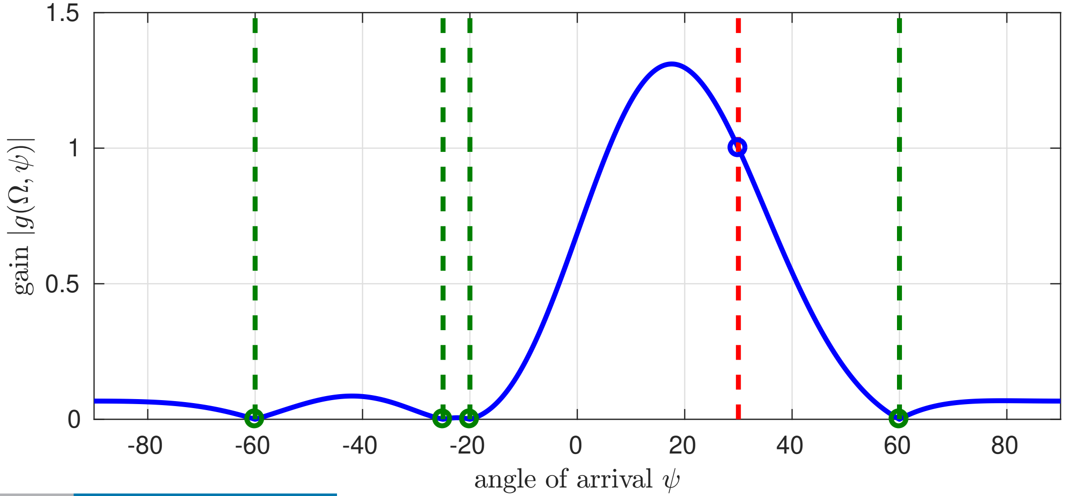

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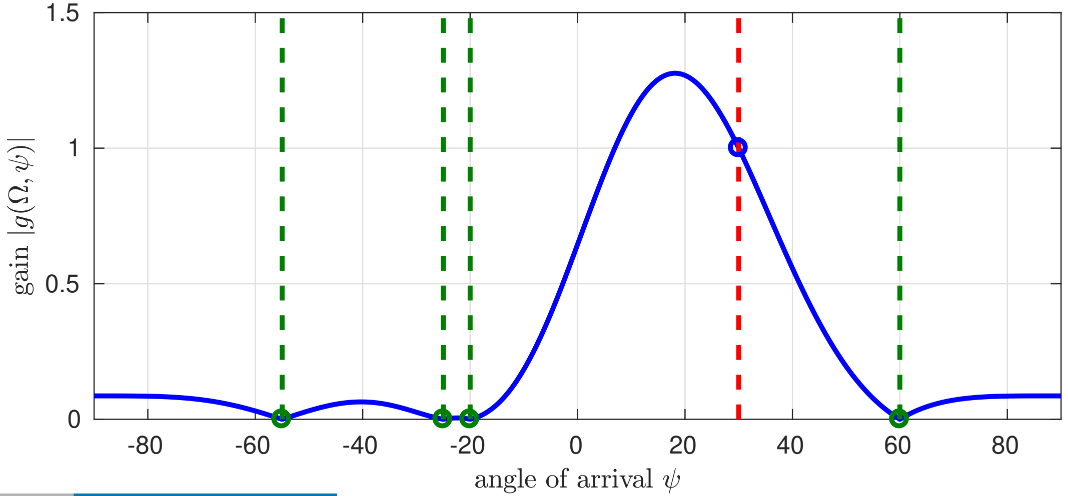

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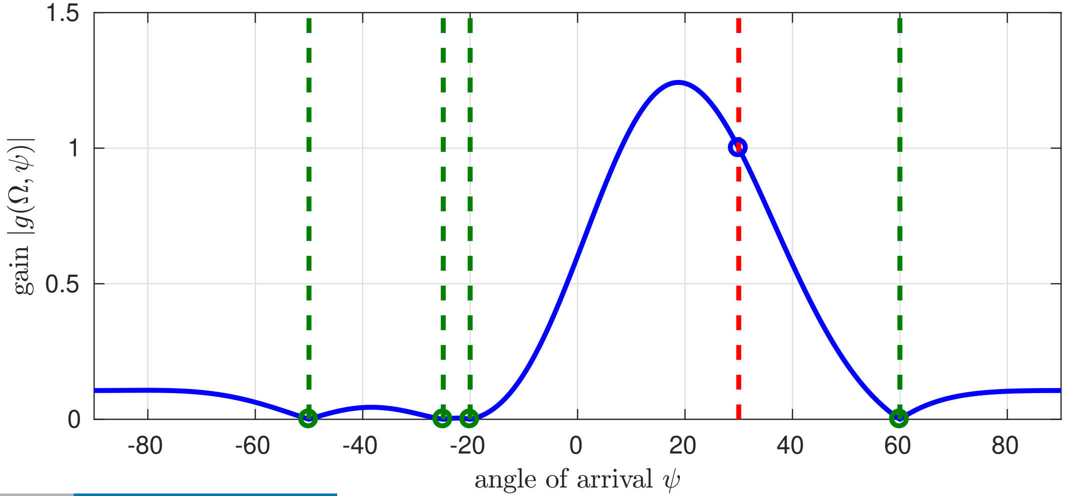
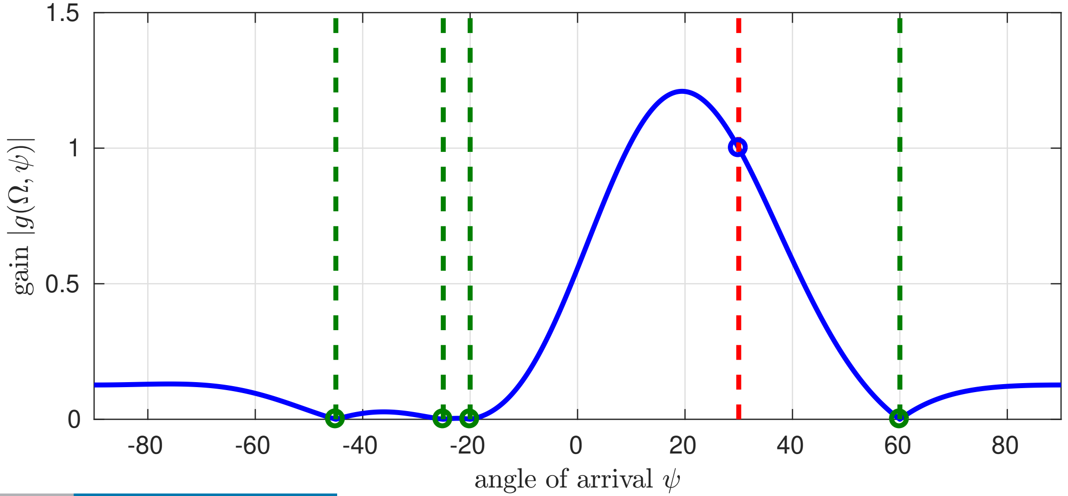


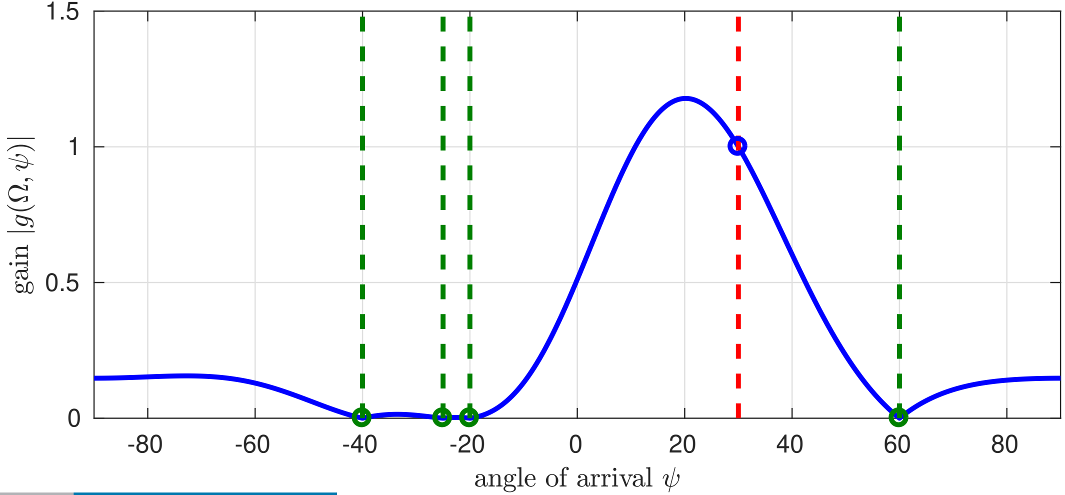
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



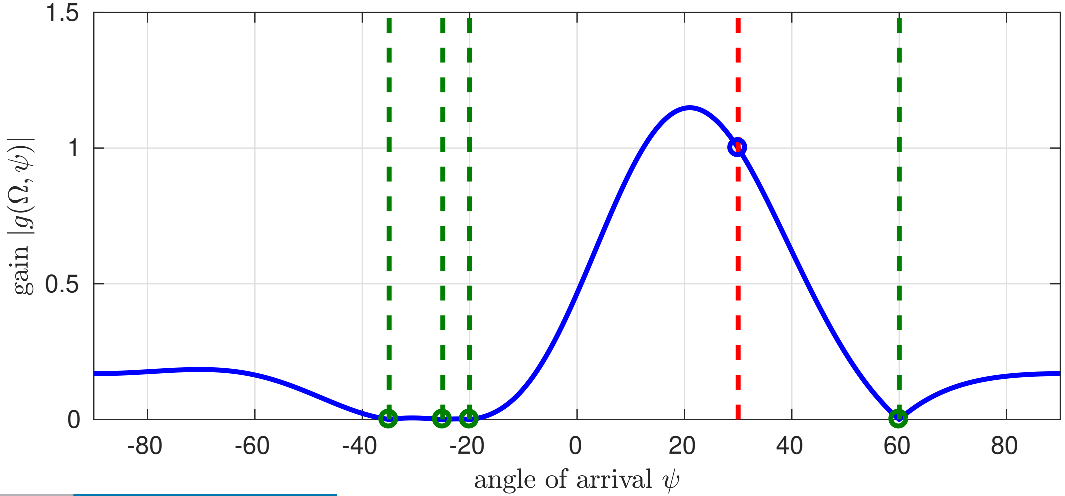
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



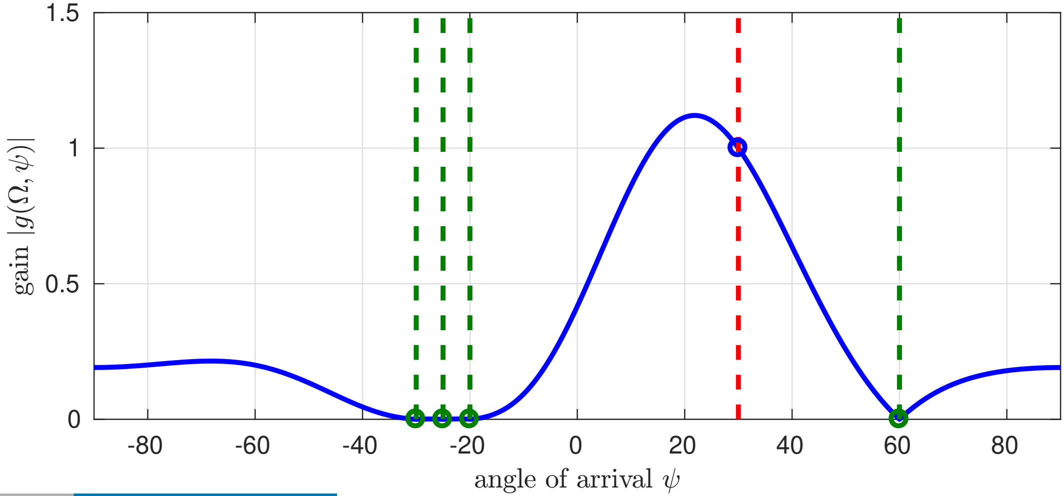
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



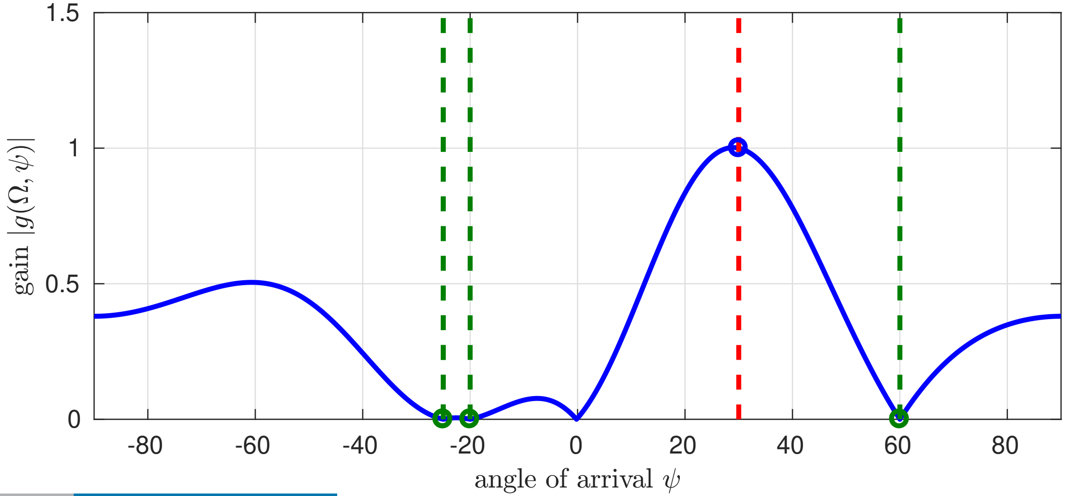
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



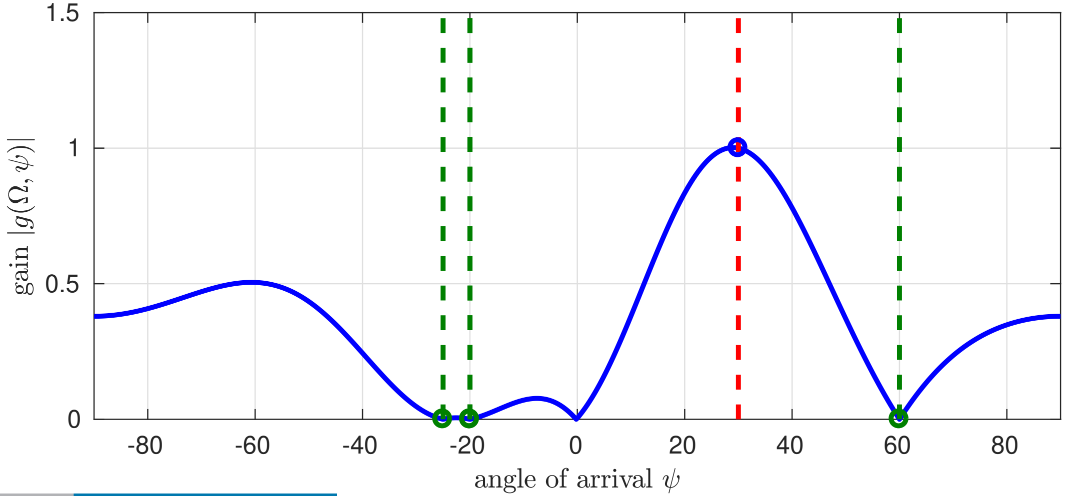
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30^\circ$, one variable and three fixed interferers:



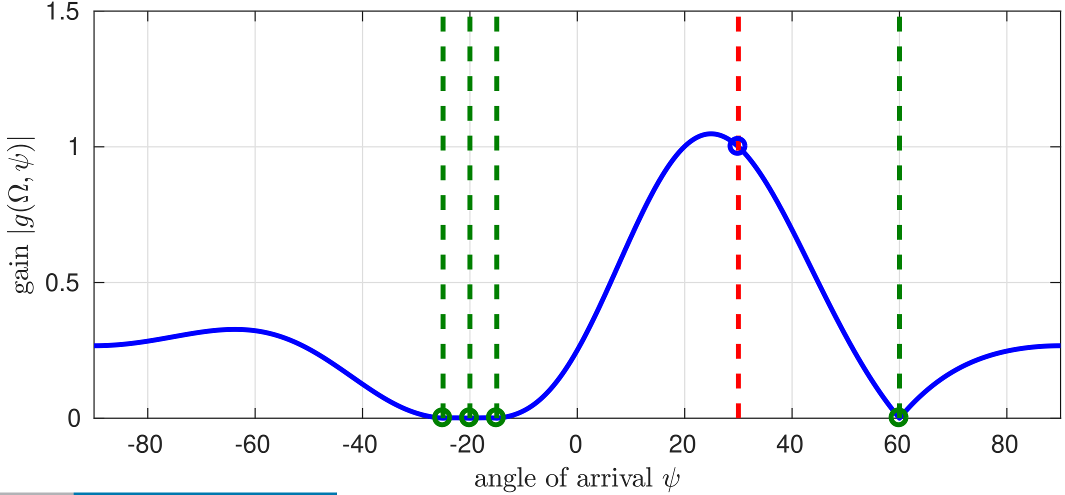
Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



Beamforming Example — Variable Interferer III

- $M = 5$ sensors, SOI $\theta_0 = 30°$, one variable and three fixed interferers:



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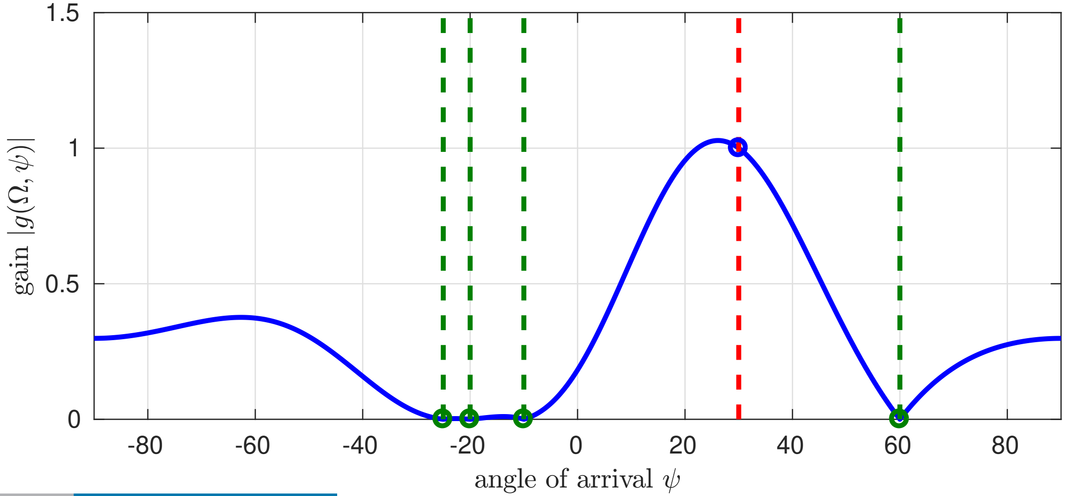

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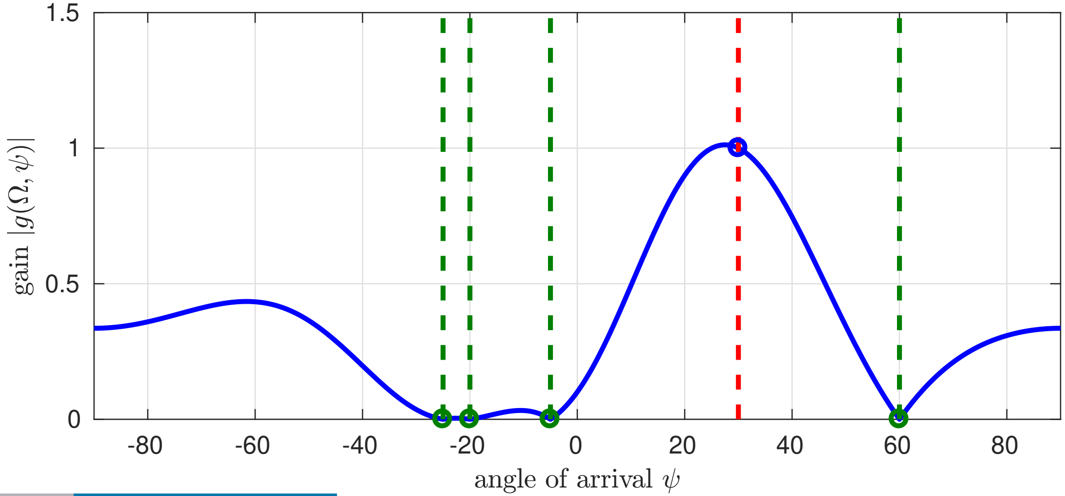

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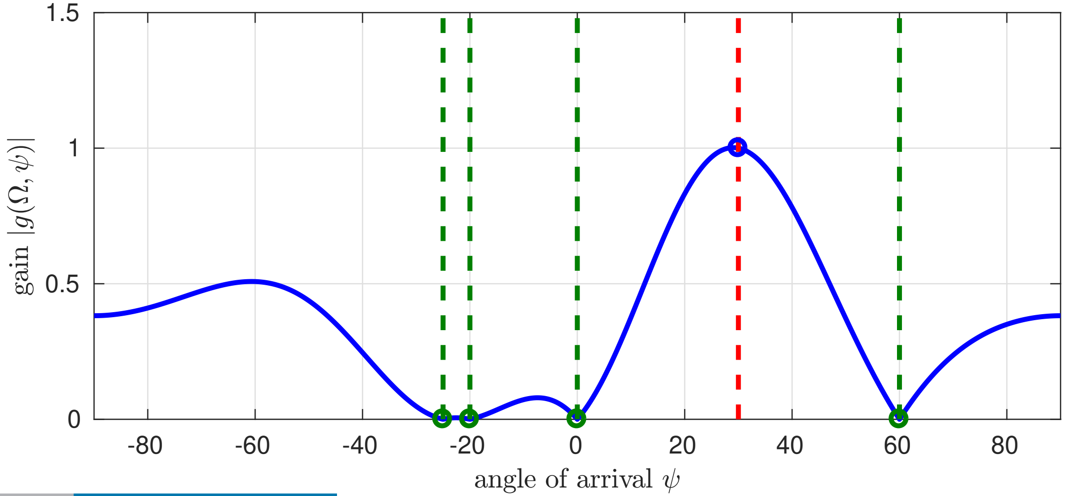

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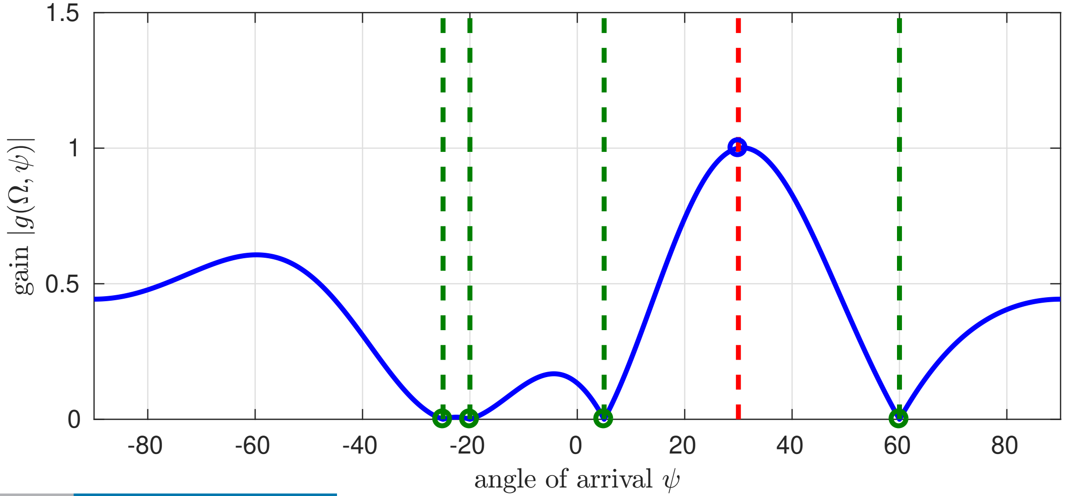

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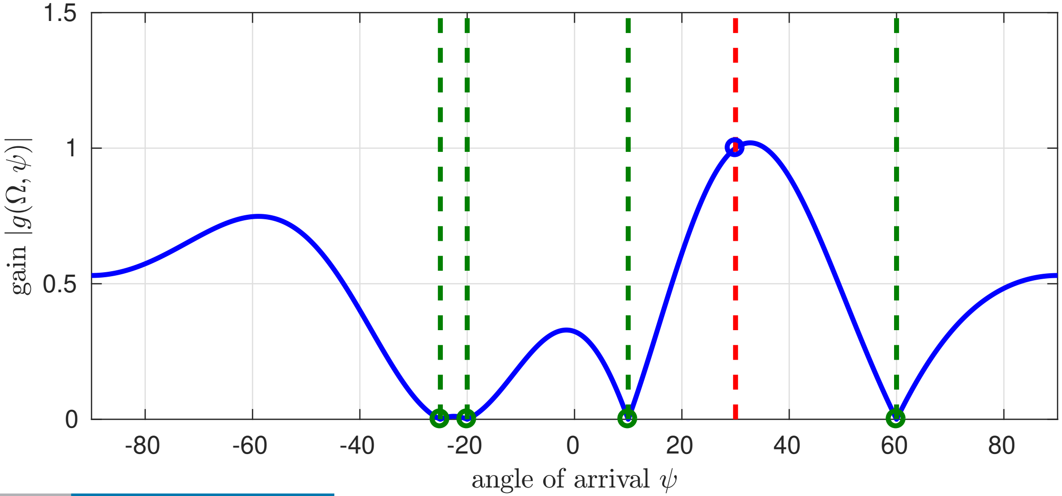

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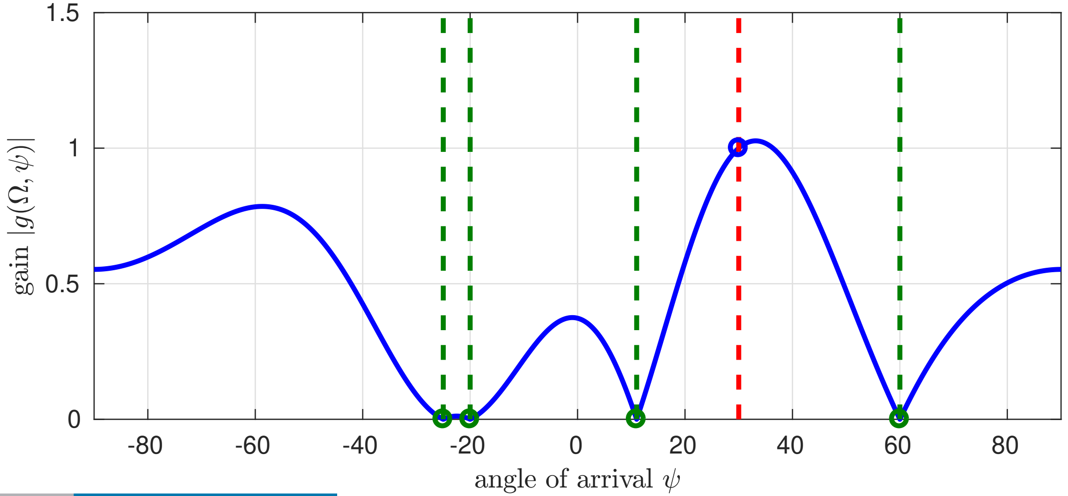

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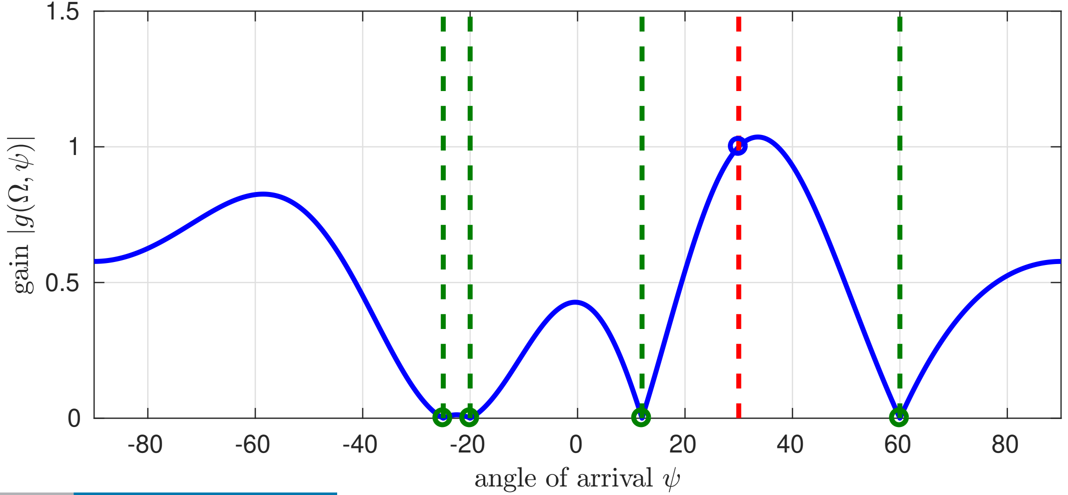

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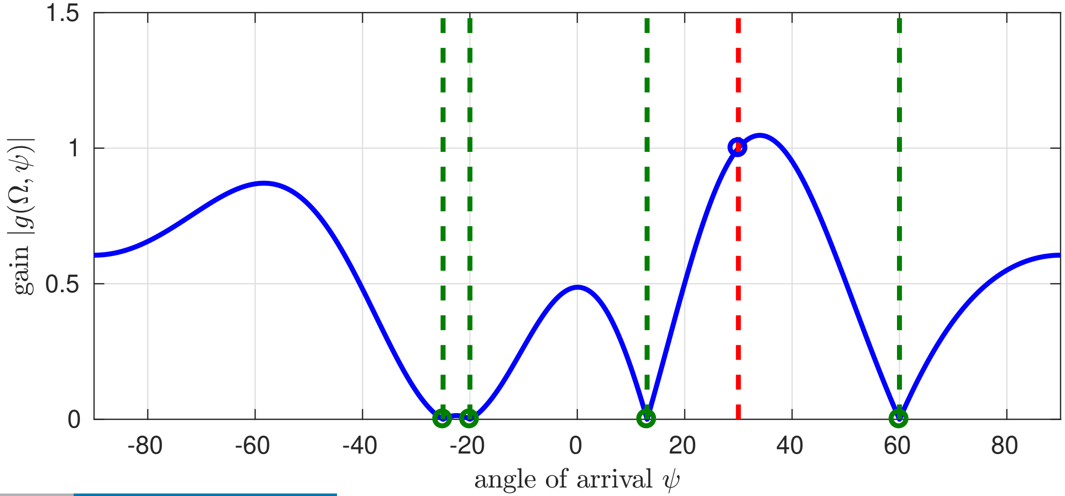

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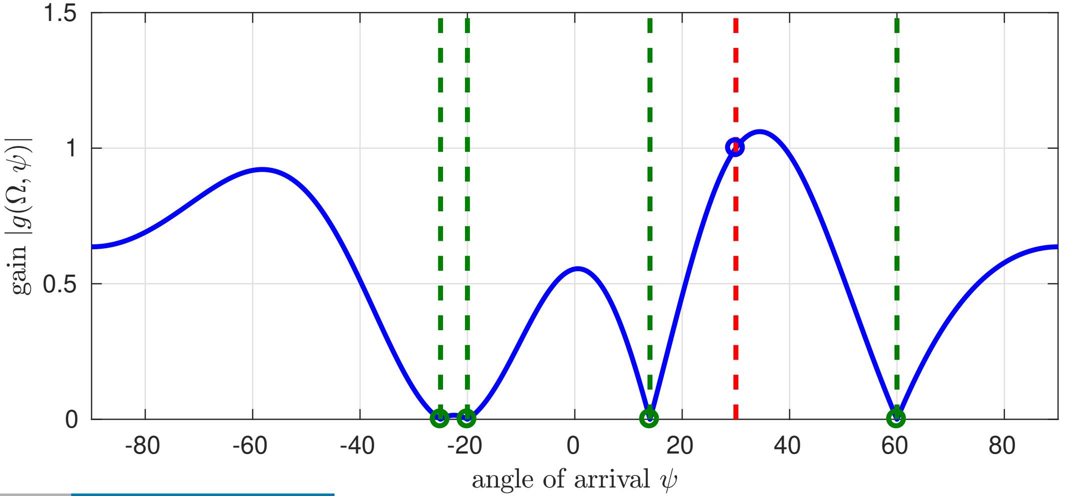

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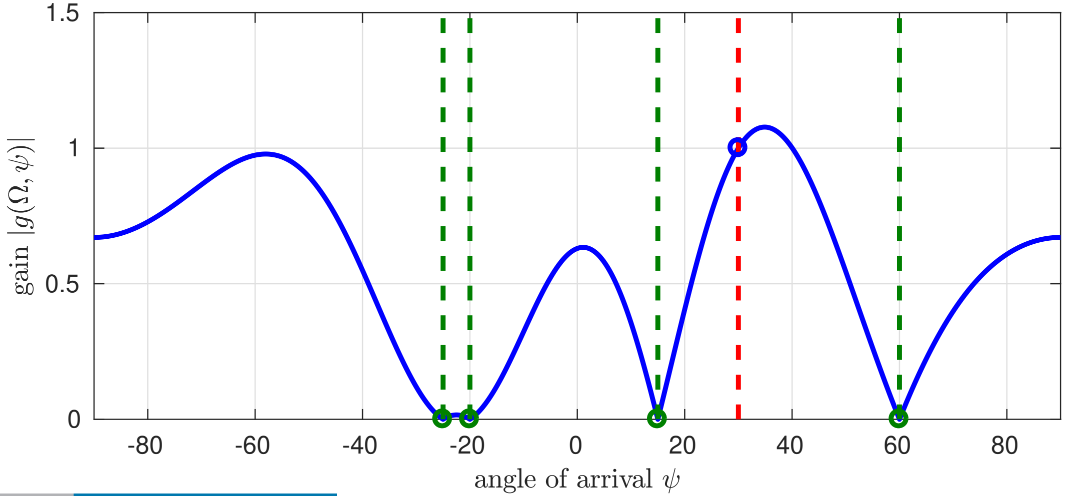

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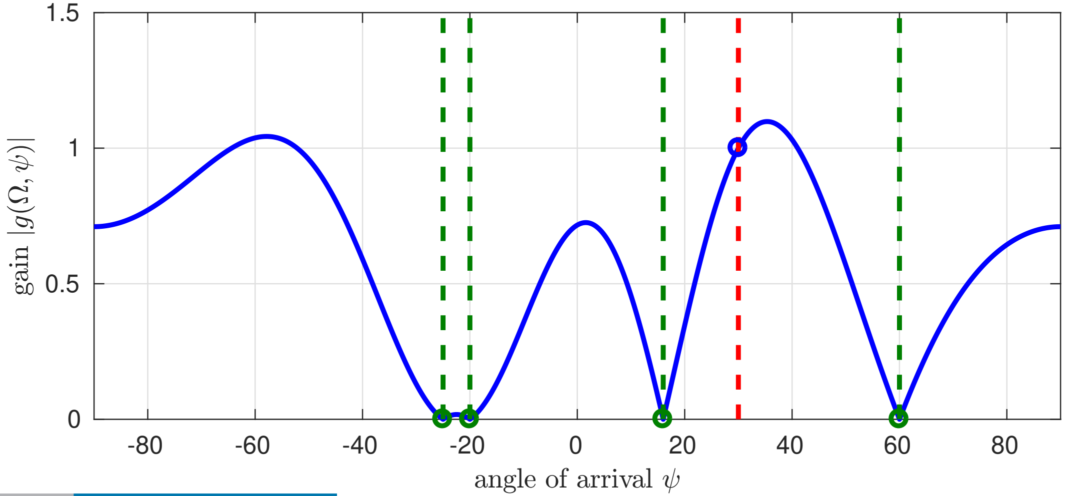

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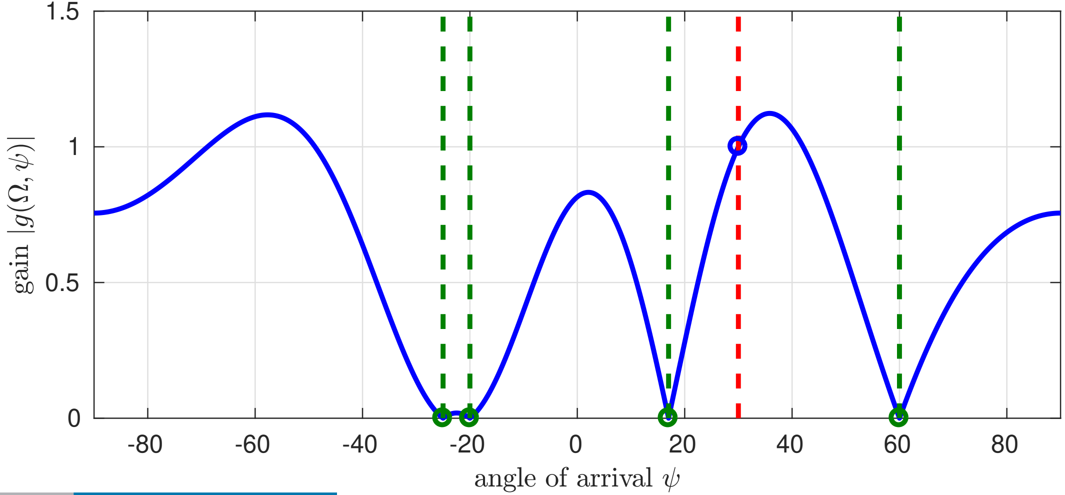

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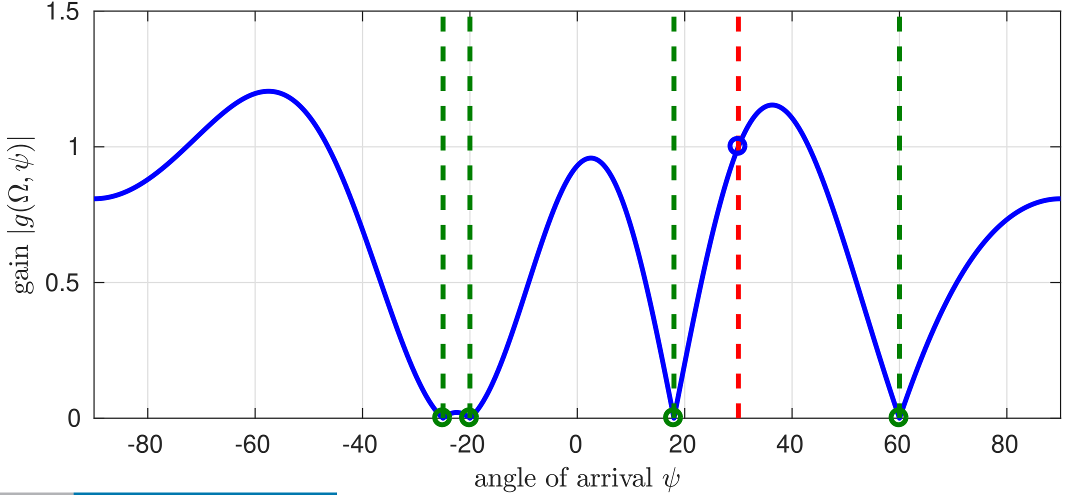

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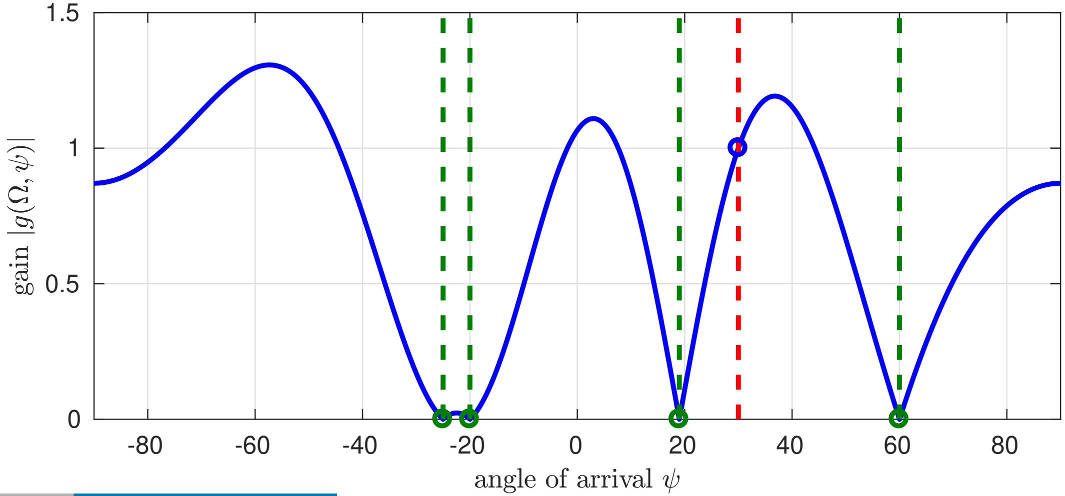

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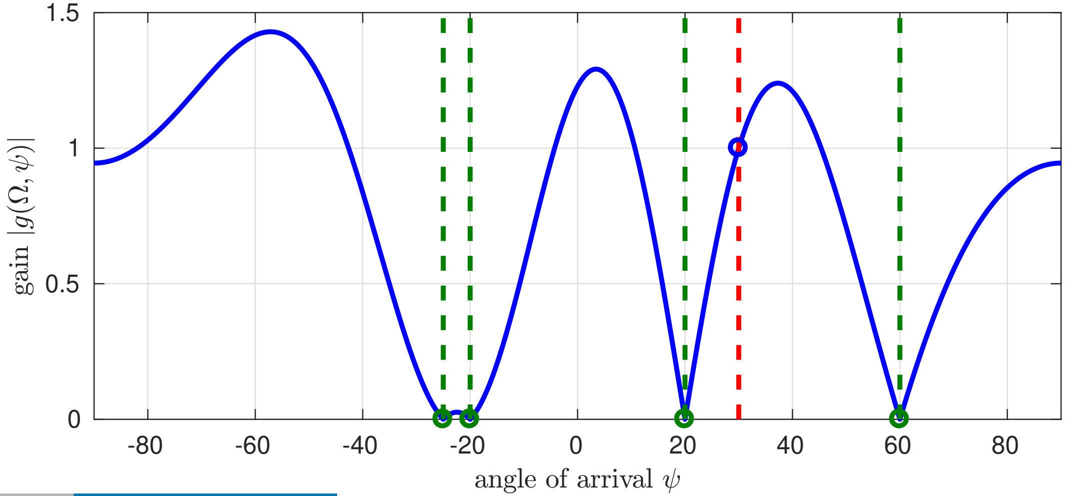

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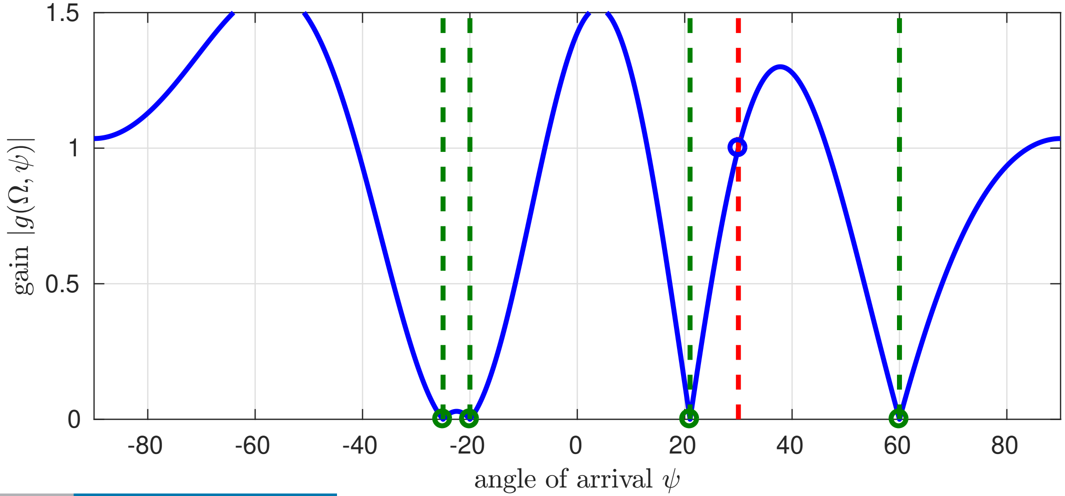

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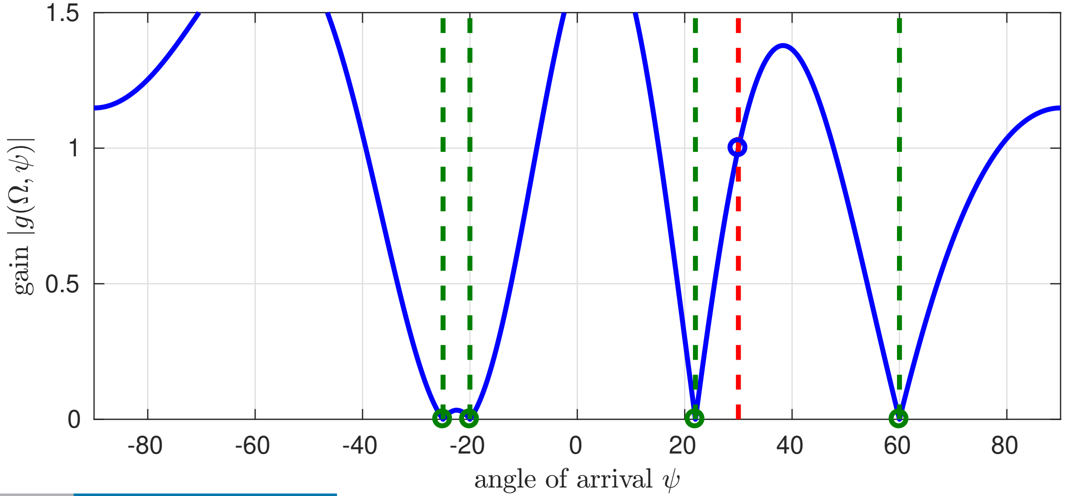

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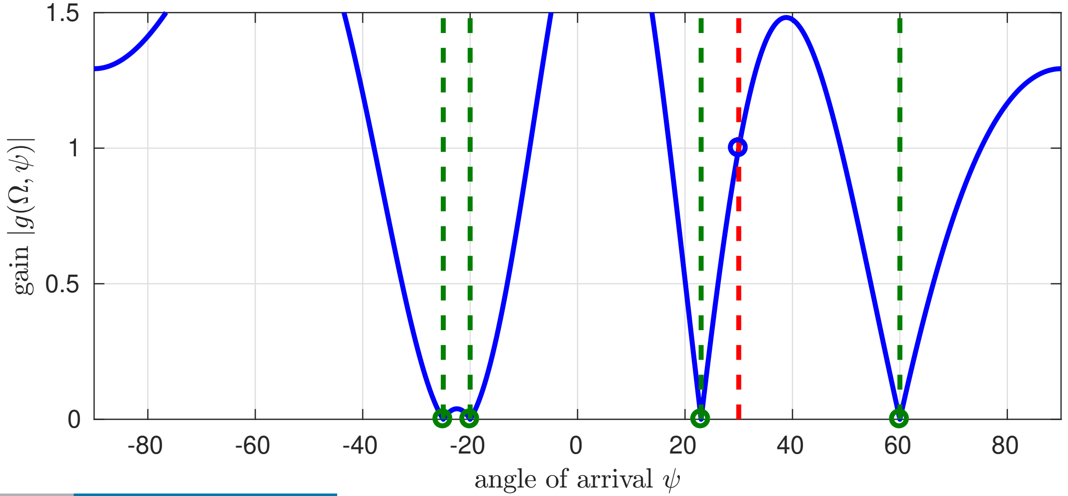

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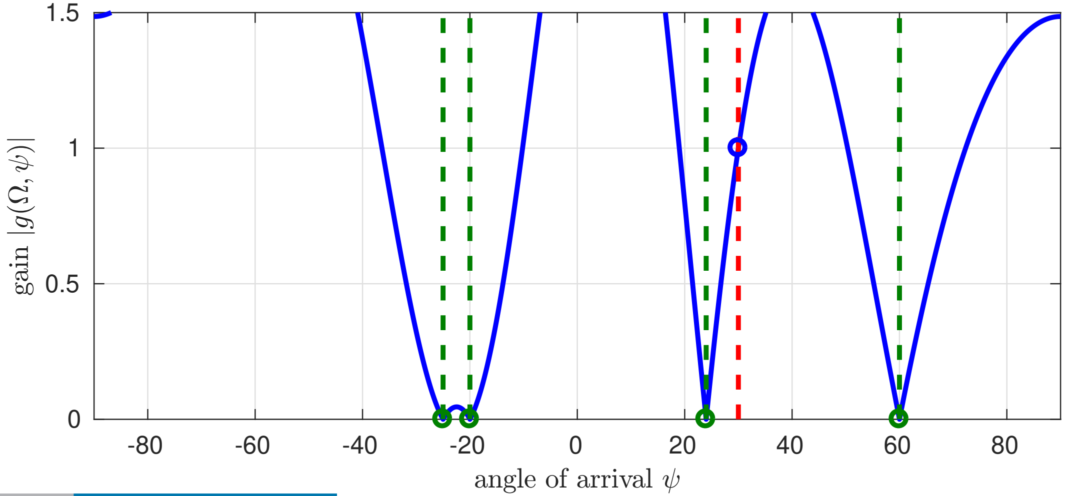

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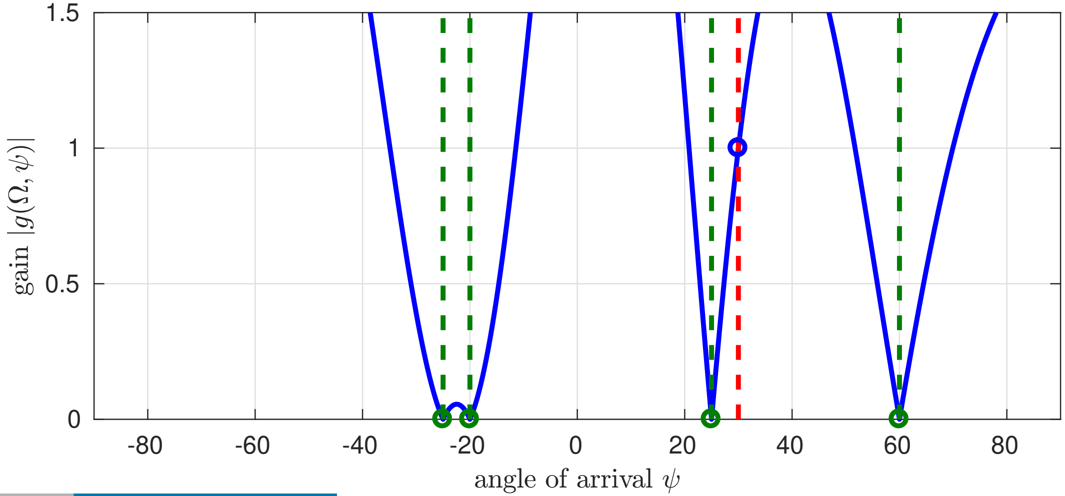
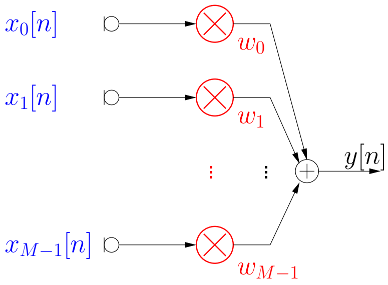


Data Independent Beamforming

- Previous beamformer designs were based on the knowledge of DoA of the signal of interest and of interfering sources;
- remaining degrees of freedom are invested to suppress spatially white noise;
- using the analogy between spatial and temporal processing, classical filter design techniques can be invoked to design arrays with a bandpass-type angular response;
- beamformers based on source parameters (frequency, DoA) rather the actual received waveforms are termed data independent beamformers;
- this is in contrast to statistically optimum beamformers, which take the received signal statistics into account.

1.8 Statistically Optimum Beamforming



- Statistically optimum beamformer minimise e.g. the signal power of the beamformer output, $y[n]$;
- to avoid the trivial solution $\mathbf{w} = \mathbf{0}$, the signal of interest needs to be protected by constraints;

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- this results in e.g. the following constrained optimisation problem

$$\min_{\mathbf{w}^*} \mathcal{E}\{|y[n]|^2\} \qquad \text{subject to} \qquad \mathbf{s}_{\Omega,\vartheta}^{\mathrm{H}} \mathbf{w} = 1$$

- the solution to this specific statistically optimum beamformer is known as the minimum variance distortionless response (MVDR).

MVDR Beamformer

- Solving the MVDR problem: minimise the power of $y[n] = \mathbf{w}^{\mathrm{H}}\mathbf{x}$ subject to the contraint $\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega_0,\vartheta_0} = 1$;
- Formulation using a Lagrange multiplier λ:

$$\frac{\partial}{\partial \mathbf{w}^*}\left(\mathbf{w}^{\mathrm{H}}\mathcal{E}\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\}\,\mathbf{w} - \lambda(\mathbf{w}^{\mathrm{H}}\mathbf{s}_{\Omega_0,\vartheta_0} - 1)\right) = \mathbf{R}_{xx}\mathbf{w} - \lambda\mathbf{s}_{\Omega_0,\vartheta_0} = \mathbf{0}$$

- the solution $\mathbf{w} = \lambda\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}$ is inserted into the constraint equation to determine λ:

$$\lambda\mathbf{s}^{\mathrm{H}}_{\Omega_0,\vartheta_0}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0} = 1$$

- therefore

$$\mathbf{w}_{\mathrm{MVDR}} = \left(\mathbf{s}^{\mathrm{H}}_{\Omega_0,\vartheta_0}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}\right)^{-1}\mathbf{R}_{xx}^{-1}\mathbf{s}_{\Omega_0,\vartheta_0}$$

- this statistically optimum beamformer has various other names, e.g. Capon beamformer.

MVDR Beamformer — Simple Case

- In the case of spatially white noise input, such that

$$\mathbf{R}_{xx} = \sigma_{xx}^2 \mathbf{I} \qquad \longrightarrow \qquad \mathbf{R}_{xx}^{-1} = \sigma_{xx}^{-2} \mathbf{I}$$

  the MVDR solution reduces to

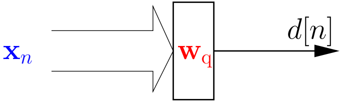
$$\mathbf{w}_{\mathrm{MVDR}} = \frac{\mathbf{s}_{\Omega_0,\vartheta_0}}{\|\mathbf{s}_{\Omega_0,\vartheta_0}\|_2^2} = \frac{\mathbf{s}_{\Omega_0,\vartheta_0}}{M} \quad ;$$

- this is identical to the data independent beamformer in the absence of interference (i.e. no spatially structured noise);

Generalised Sidelobe Canceller (GSC)

- The generalised sidelobe canceller is a specific implementation of the MVDR beamformer; it transforms the constrained MVDR problem into an unconstrained optimisation problem;
- a first guess at the solution is performed by the quiescent beamformer $\mathbf{w}_{\mathrm{q}}$, which is identical to the previously defined data independent beamformer, obtained by inverting the constraint equation

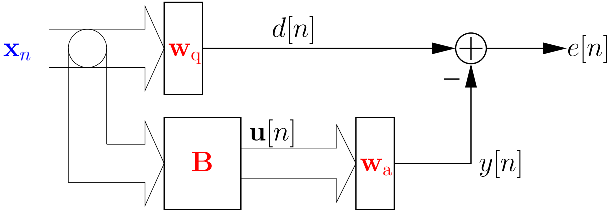
$$\mathbf{C}^{\mathrm{H}}\mathbf{w}_{\mathrm{q}} = \mathbf{f} \qquad \longrightarrow \qquad \mathbf{w}_{\mathrm{q}} = \left(\mathbf{C}^{\mathrm{H}}\right)^{\dagger}\mathbf{f}$$



- the quiescent beamformer eliminates interferers captured by $\mathbf{C}$ and $\mathbf{f}$, but passes the signal of interest, any interferers unaccounted for in the constraints, and spatially distributed noise.

GSC — Idea

- GSE idea: produce array signals that are free of any contribution from the signal of interest, and use the resulting signal vector $\mathbf{u}[n]$ to eliminate remaining interference from the quiescent output:



- the blocking matrix $\mathbf{B}$ eliminates the signal of interest and any interferers captured by the constraints;
- the vector $\mathbf{w}_\mathrm{a}$ will be based on the statistics of $\mathbf{u}[n]$ and $d[n]$ to minimise the beamformer output variance $\mathcal{E}\{|e[n]|^2\}$.

GSC — Blocking Matrix

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- In order to project away from the constraints,

$$\mathbf{B}\cdot\mathbf{C} = \mathbf{B}\cdot\left[\mathbf{s}_{\Omega_0,\vartheta_0}\ \ \mathbf{s}_{\Omega_1,\vartheta_1}\ \ \dots\ \ \mathbf{s}_{\Omega_{r-1},\vartheta_{r-1}}\right] \;=\; \mathbf{0}$$

- assuming that the r constraints are linearly independent, the singular value decomposition of the constraint matrix yields

$$\mathbf{B}\cdot\left[\mathbf{U}_0\ \ \mathbf{U}_0^{\perp}\right]\left[\begin{array}{ccc|c} \sigma_0 & & & \\ & \ddots & & \mathbf{0} \\ & & \sigma_{r-1} & \\ \hline & \mathbf{0} & & \mathbf{0} \end{array}\right]\cdot\mathbf{V}^{\mathrm{H}} \;=\; \mathbf{0}$$

- the matrix $\mathbf{U}_0^{\perp}\in\mathbb{C}^{M\times(M-r)}$ spans the nullspace of $\mathbf{C}^{\mathrm{H}}$, and

$$\mathbf{B} = (\mathbf{U}_0^{\perp})^{\mathrm{H}} \in \mathbb{C}^{(M-r)\times M}$$

has the required property, as $(\mathbf{U}_0^{\perp})^{\mathrm{H}}\cdot\left[\mathbf{U}_0\ \ \mathbf{U}_0^{\perp}\right]\boldsymbol{\Sigma} = [\mathbf{0}\ \ \mathbf{I}]\cdot\boldsymbol{\Sigma} = \mathbf{0}$.

GSC — Unconstrained Optimisation

- The beamforming vector $\mathbf{w}_\mathrm{a}$ is adjusted to minimise the output power;
- the MMSE or Wiener solution is given by

$$\mathbf{w}_\mathrm{a} = \mathbf{R}_{uu}^{-1} \cdot \mathbf{p} = \frac{\mathbf{B}\mathbf{R}_{xx}(\mathbf{C}^\mathrm{H})^\dagger \mathbf{f}}{\mathbf{B}\mathbf{R}_{xx}\mathbf{B}^\mathrm{H}}$$

with the covariance matrix

$$\mathbf{R}_{uu} = \mathcal{E}\left\{\mathbf{u}[n] \cdot \mathbf{u}^\mathrm{H}[n]\right\} = \mathbf{B}\,\mathcal{E}\left\{\mathbf{x}[n] \cdot \mathbf{x}^\mathrm{H}[n]\right\}\,\mathbf{B}^\mathrm{H} = \mathbf{B}\mathbf{R}_{xx}\mathbf{B}^\mathrm{H}$$

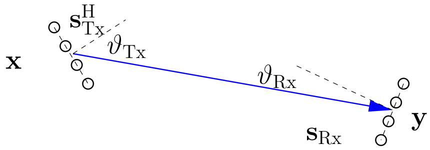
and the cross-correlation vector

$$\mathbf{p} = \mathcal{E}\{\mathbf{u}[n] \cdot d^*[n]\} = \mathbf{B}\mathbf{R}_{xx}\mathbf{w}_\mathrm{q}$$

- iterative optimisation schemes, such as the least mean squares (LMS) algorithm may be used to approximate the MMSE solution.

1.9 Beamforming and MIMO Processing

- Assume a transmission scenario with an M-element transmit (Tx) antenna array and an N-element receive (Rx) array;



- in the absence of scatterers and any attenuation, the farfield transmission from the transmit antenna is characterised by a steering vector $\mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$;
- the incoming waveform at the Rx device is described by another steering vector $\mathbf{s}_{\mathrm{Rx}}$;
- the overall MIMO system between a Tx vector $\mathbf{x} \in \mathbb{C}^M$ and an Rx vector $\mathbf{y} \in \mathbb{C}^N$ is described as

$$\mathbf{y} = \mathbf{s}_{\mathrm{Rx}} \cdot \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}} \cdot \mathbf{x} \;=\; \mathbf{H}\mathbf{x}$$

- the MIMO system matrix $\mathbf{H} = \mathbf{s}_{\mathrm{Rx}} \cdot \mathbf{s}_{\mathrm{Tx}}^{\mathrm{H}}$ is rank one only.

MIMO Requirements

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- The farfield assumption is convenient for beamforming, but leads to a rank one MIMO system matrix which is incompatible with the desire to extract multiple independent subchannels or with to achieve diversity;
- rich scattering in connection with MIMO usually implies multiple reflections of signals;
- together with a sufficiently large antenna spacing means that the farfield assumption is invalid and the MIMO system matrix is not rank deficient;
- some suggestions of “sufficiently large spacing” imply an antenna element distance of $d > 10\lambda$;
- recall spatial sampling requires $d < \frac{1}{2}\lambda$!

Beamforming with Spatial Aliasing

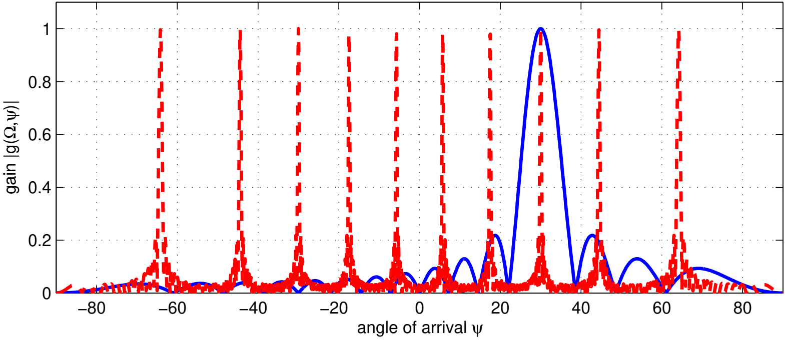
- For a flexible spatial sampling with $d = \alpha\lambda$, $0 < \alpha \in \mathbb{R}$, the steering vector for a waveform with normalised angular frequency Ω and DoA ϑ is

$$\mathbf{y} = e^{j\Omega n} \begin{bmatrix} 1 \\ e^{j2\alpha\Omega\sin(\vartheta)} \\ \vdots \\ e^{j2\alpha(M-1)\Omega\sin(\vartheta)} \end{bmatrix} = \mathbf{s}_{2\alpha\Omega,\vartheta} \cdot e^{j\Omega}$$

- inspecting $\mathbf{s}_{2\alpha\Omega,\vartheta}$ the steering vector is aliased to a different frequency $2\alpha\Omega$;
- although the correct frequency can be retrieved unambigiously from temporal sampling of any array element, at Ω various different angles could provide the same steering vector $\mathbf{s}_{2\alpha\Omega,\vartheta}$;
- the array performs spatial undersampling, resulting in spatial aliasing.

Spatial Undersampling Example

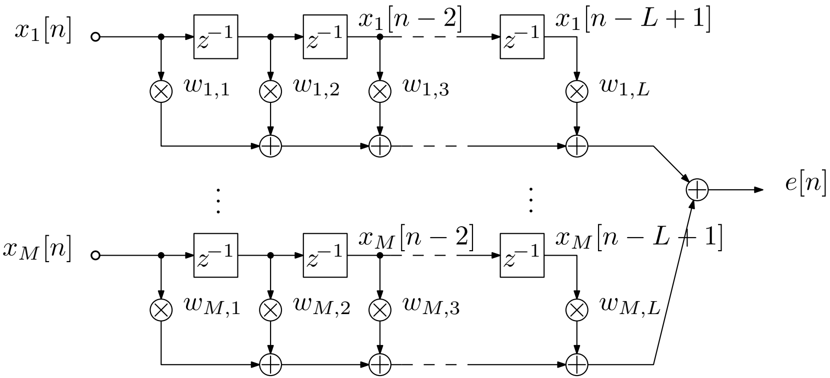
- Beamforming parameters: signal of interest with $\Omega = \frac{\pi}{2}$, direction of arrival $\vartheta = 30°$, $M = 32$ array elements;
- data independent beamformer design with correct spatial sampling ($d = \lambda/2$) and incorrect spatial sampling ($d = 10\lambda$):



- MIMO systems perform beamforming, but may dissipate energy into aliased directions.

1.10 Broadband MVDR Beamformer

- Each sensor is followed by a tap delay line of dimension L, giving a total of ML coefficients in a vector $\mathbf{v} \in \mathbb{C}^{ML}$



Broadband MVDR Beamformer Constraints

- A larger input vector $\mathbf{x}_n \in \mathbb{C}^{ML}$ is generated, also including lags;
- the general approach is similar to the narrowband system, minimising the power of $e[n] = \mathbf{v}^{\mathrm{H}}\mathbf{x}_n$;
- however, we require several constraint equations to protect the signal of interest, e.g.

$$\mathbf{C} = [\mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_0), \;\; \mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_1) \;\; \ldots \;\; \mathbf{s}(\vartheta_{\mathrm{s}}, \Omega_{L-1})] \tag{1}$$

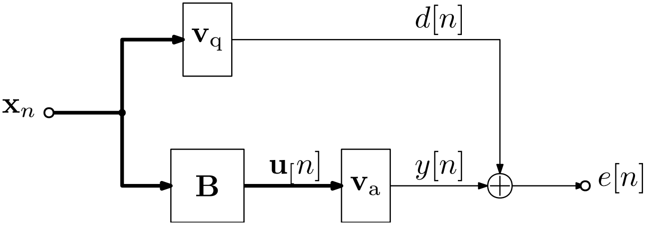
- these L constraints pin down the response to unit gain at L separate points in frequency:

$$\mathbf{C}^{\mathrm{H}}\mathbf{v} = \mathbf{1} \;; \tag{2}$$

- generally $\mathbf{C} \in \mathbb{C}^{ML \times L}$, but simplifications can be applied if the look direction is towards broadside.

Broadband Generalised Sidelobe Canceller

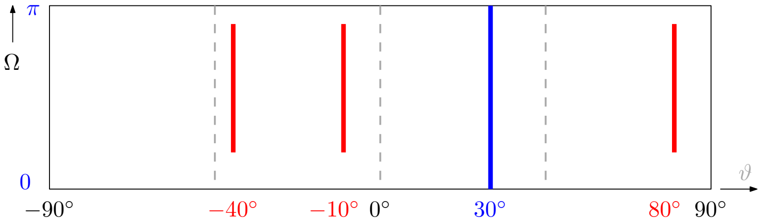
- A quiescent beamformer $\mathbf{v}_\mathrm{q} = \left(\mathbf{C}^\mathrm{H}\right)^\dagger \mathbf{1} \in \mathbb{C}^{ML}$ picks the signal of interest;
- the quiescent beamformer is optimal for AWGN but generally passes structured interference;
- the output of the blocking matrix $\mathbf{B}$ contains interference only, which requires $[\mathbf{BC}]$ to be unitary; hence $\mathbf{B} \in \mathbb{C}^{ML\times(M-1)L}$;
- an adaptive noise canceller $\mathbf{v}_\mathrm{a} \in \mathbb{C}^{(M-1)L}$ aims to remove the residual interference:



- note: all dimensions are determined by $\{M, L\}$.

Broadband Beamformer Example

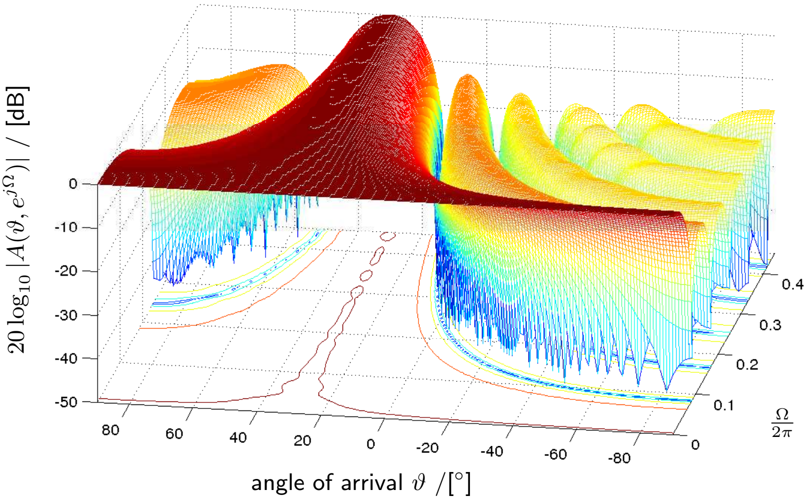
- We assume a signal of interest from $\vartheta = 30°$;
- three interferers with angles $\vartheta_i \in \{-40°, -10°, 80°\}$ active over the frequency range $\Omega = 2\pi \cdot [0.1;\ 0.45]$ at signal to interference ratio of -40 dB;



- $M = 8$ element linear uniform array is also corrupted by spatially and temporally white additive Gaussian noise at 20 dB SNR;
- tap-delay-line length: $L = 150$;
- cost per iteration: approx. 2 MMACs (standard), can be reduced to 10 kMACs when efficiently implemented.

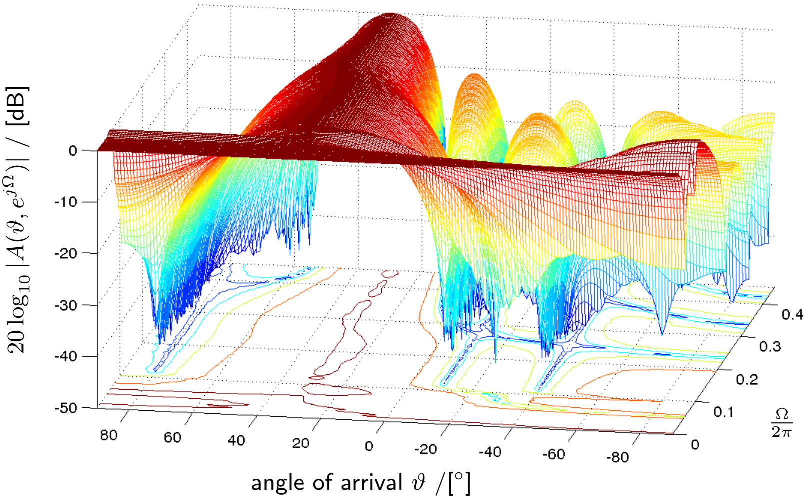
Broadband Quiescent Beamformer

- Directivity pattern of quiescent standard broadband beamformer:



Optimised Broadband Beamformer

- Directivity pattern of the broadband beamformer:



1.11 Summary

- Spatial sampling by an array of sensors (e.g. antenna elements) has been explored;
- the spatial data window of a narrowband source is characterised by the steering vector;
- appropriate data independent beamformers can be designed based on the steering vectors of desired sources and interferers;
- statistically optimum beamformers are based on the signal statistics;
- a specific statistically optimum beamformer, the generalised sidelobe canceller, has been reviewed — it uses signal statistics to improve the performance of a data independent beamformer derived from the constraint equations;
- some similarities and differences between beamforming and MIMO systems have been highlighted;
- broadband beamforming requires the inclusion of tap delay lines.