Decision Theory

João Mota

UDRC Summer School, 2023

Heriot-Watt University

Decision theory

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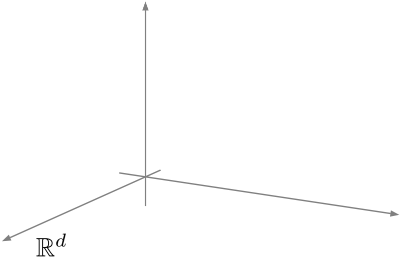
Problem

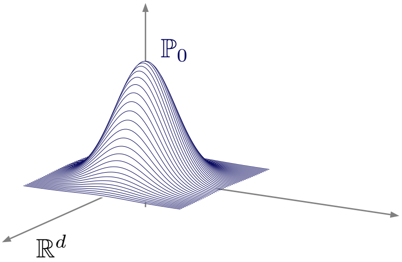
Decision theory



Problem

We observe $\boldsymbol{X} \in \mathbb{R}^d$

Decision theory

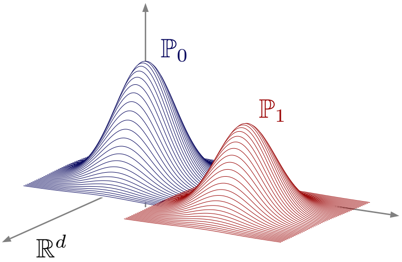


Problem

We observe $\boldsymbol{X} \in \mathbb{R}^d$

$\boldsymbol{X} \sim \mathbb{P}_0$

Decision theory



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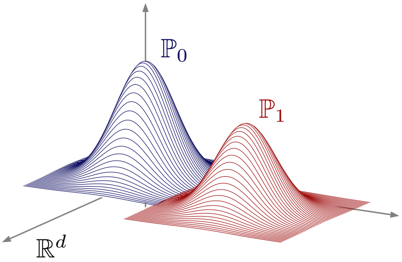
$\boldsymbol{X} \sim \mathbb{P}_0$ or $\boldsymbol{X} \sim \mathbb{P}_1$?

Decision theory



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In classical *decision theory*, we *know* the distributions $\mathbb{P}_0$ and $\mathbb{P}_1$

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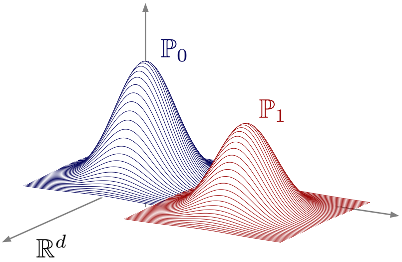

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$\boldsymbol{X} \sim \mathbb{P}_0$ or $\boldsymbol{X} \sim \mathbb{P}_1$?

In classical *decision theory*, we *know* the distributions $\mathbb{P}_0$ and $\mathbb{P}_1$

In *machine learning*, we *have to estimate* $\mathbb{P}_0$ and $\mathbb{P}_1$ from data

Example in 1D: Spam

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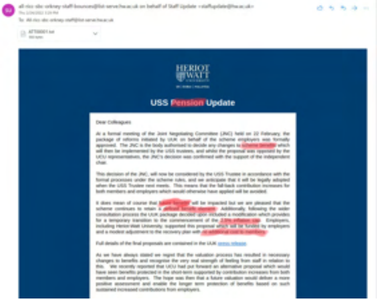

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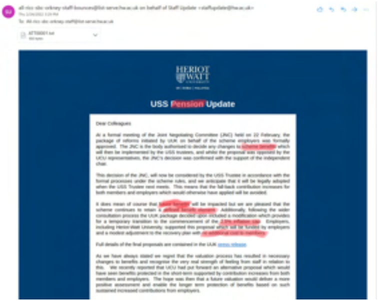

$X \in \mathbb{R}$: number of **spam words** in a message

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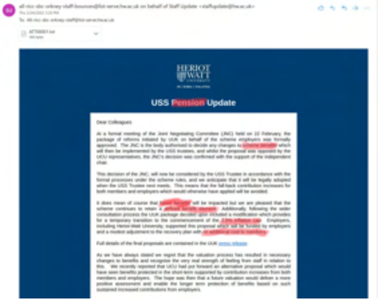

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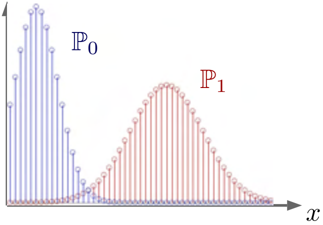

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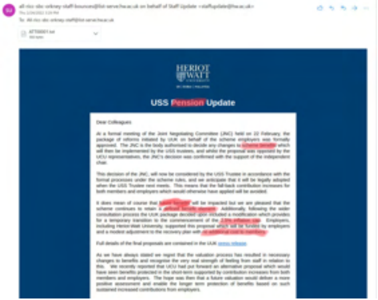

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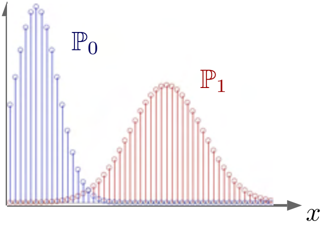

Null Hypothesis

H_0: message isn't spam

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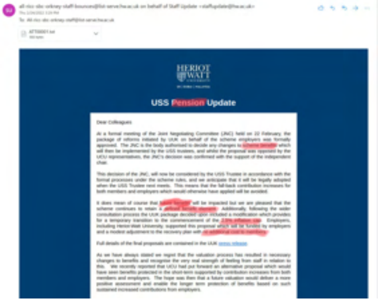


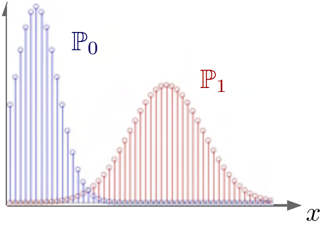
$X \in \mathbb{R}$: number of **spam words** in a message

Null Hypothesis

H_0: message isn't spam

Alternative Hypothesis

H_1: message is spam



Signal vs Noise

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Consider a test for detecting:

if given email is spam

Signal vs Noise

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if given email is spam   presence of aircraft in radar

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if given email is spam	presence of aircraft in radar
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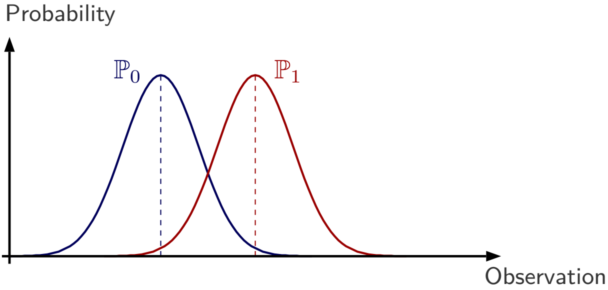

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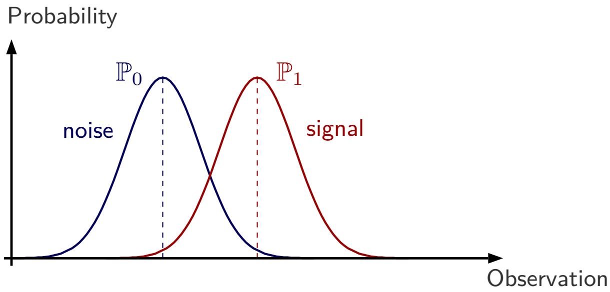

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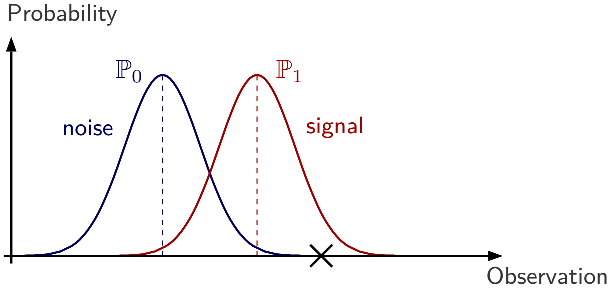

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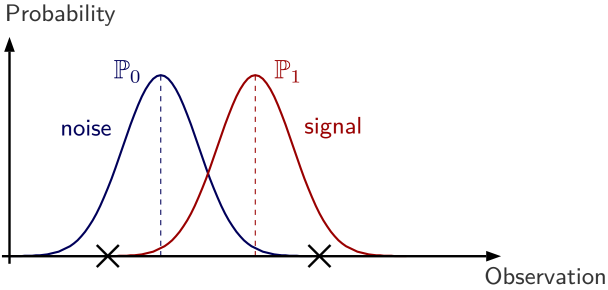

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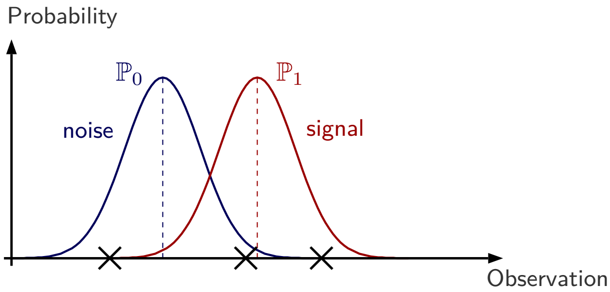

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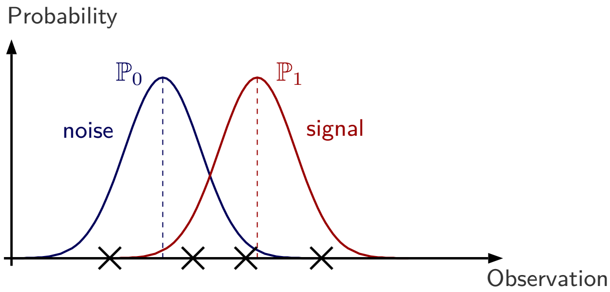

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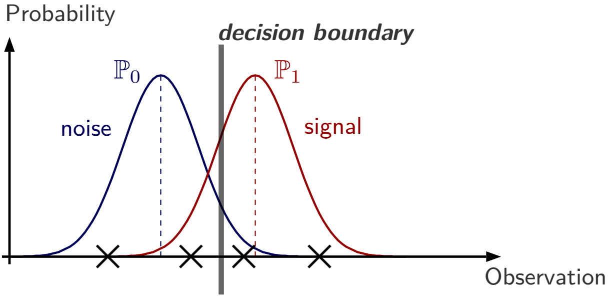

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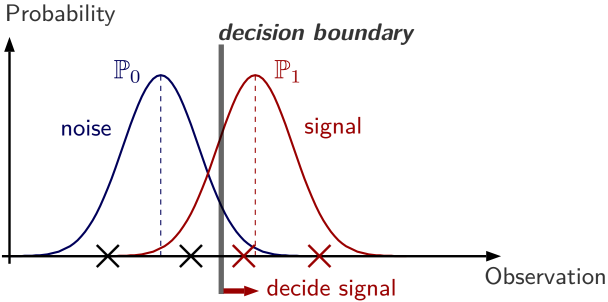

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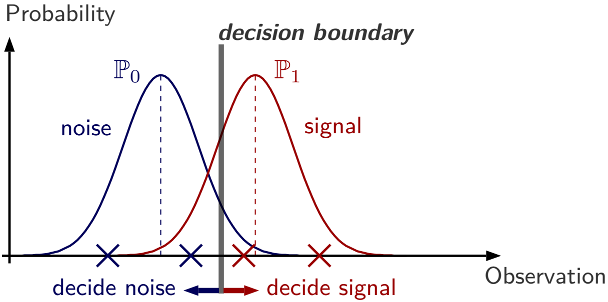

The Decision Tradeoff

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True hypothesis	Decide noise	Decide signal
noise		
signal		

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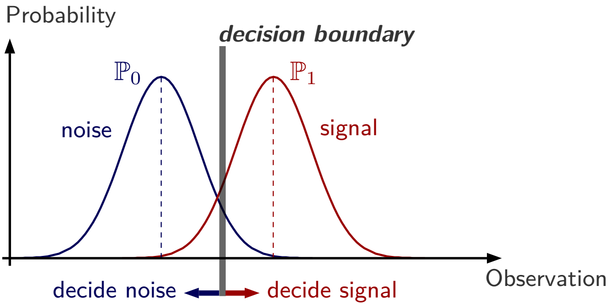
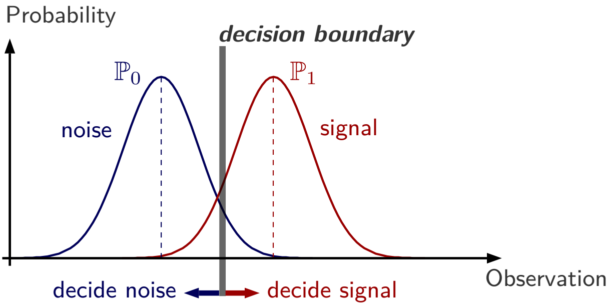


The Decision Tradeoff

True hypothesis	Decide noise	Decide signal
noise	✓	
signal		



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True hypothesis	Decide noise	Decide signal
noise	✓	
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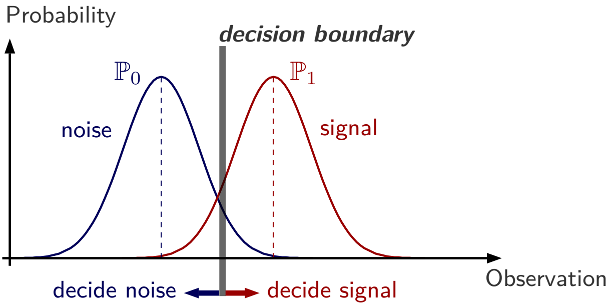

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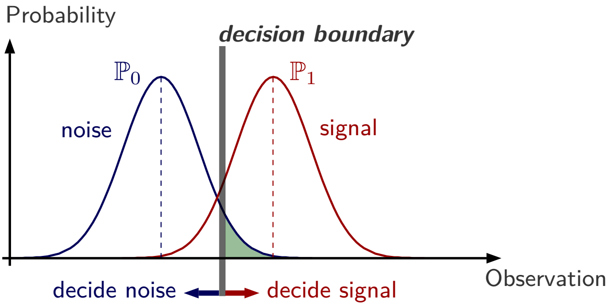
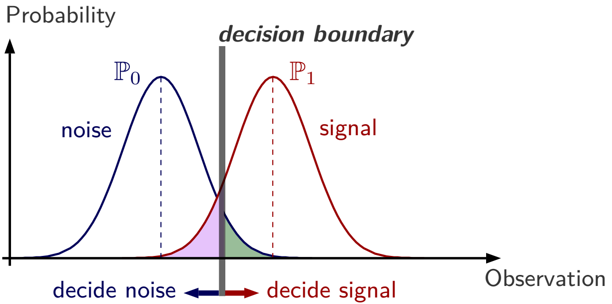
True hypothesis	Decide noise	Decide signal
noise	✓	false alarm
signal		✓



The Decision Tradeoff

True hypothesis	Decide noise	Decide signal
noise	✓	false alarm
signal	missed detection	✓



Improving the Tradeoff

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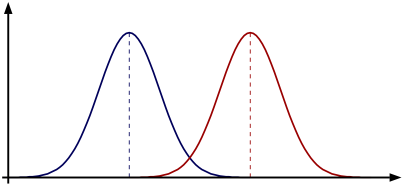
Larger effect size



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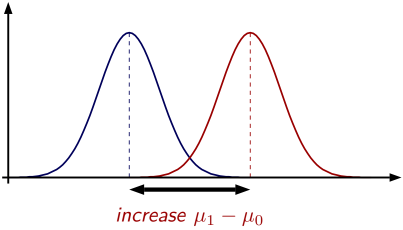

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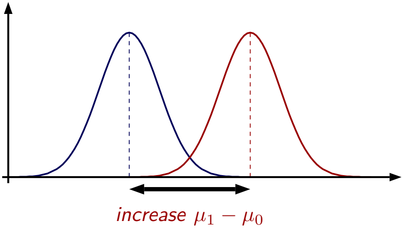
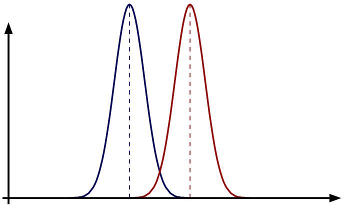


Better/more measurements



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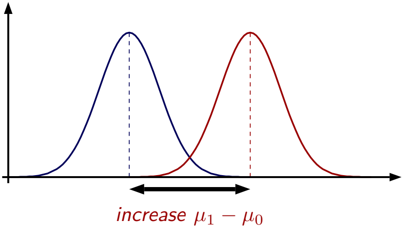

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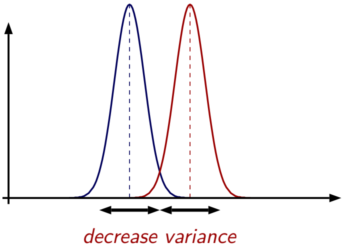
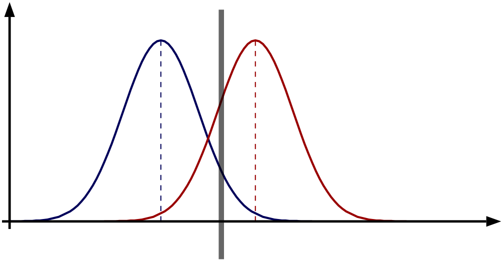
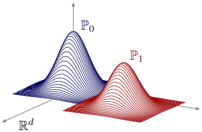


Where to place the decision boundary?



Decision and loss functions



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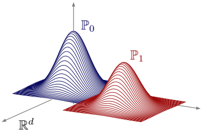
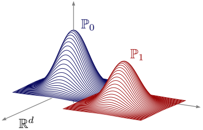


True label

$$Y = \begin{cases} 0 & , \text{ if } H_0 \text{ is true} \\ 1 & , \text{ if } H_1 \text{ true} \end{cases}$$

Decision and loss functions



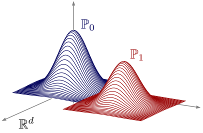
True label

$$Y = \begin{cases} 0 & , \text{ if } {\color{blue}H_0} \text{ is true} \\ 1 & , \text{ if } {\color{red}H_1} \text{ true} \end{cases}$$

Decision function $f : \mathbb{R}^d \to \{0, 1\}$

$$f(\boldsymbol{X}) = \begin{cases} 0 & , \text{ if we } \underline{\textit{decide}} \ {\color{blue}H_0} \\ 1 & , \text{ if we } \underline{\textit{decide}} \ {\color{red}H_1} \end{cases}$$

Decision and loss functions



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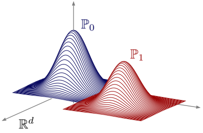
Loss function $\ell : \{0, 1\} \times \{0, 1\} \to \mathbb{R}$

Decision and loss functions



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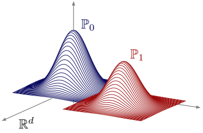

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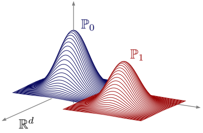

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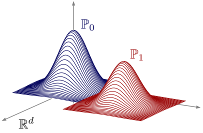

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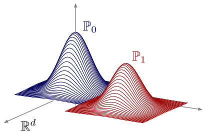

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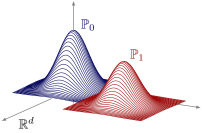

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Risk and Optimal Decision

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Given *decision function* $f : \mathbb{R}^d \to \{0, 1\}$ and *loss* $\ell : \{0, 1\}^2 \to \mathbb{R}$,

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Risk:

Risk and Optimal Decision

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Risk: $$\boxed{R[f] := \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]}$$

Risk and Optimal Decision

Given *decision function* $f : \mathbb{R}^d \to \{0,1\}$ and *loss* $\ell : \{0,1\}^2 \to \mathbb{R}$,

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$$R[f] := \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

where $\mathbb{E}_{\boldsymbol{X}Y}[\cdot]$ is the expectation with respect to $\boldsymbol{X}$ and Y

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Given *decision function* $f : \mathbb{R}^d \to \{0,1\}$ and *loss* $\ell : \{0,1\}^2 \to \mathbb{R}$,

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Optimal decision problem:

Risk and Optimal Decision

Given *decision function* $f : \mathbb{R}^d \to \{0,1\}$ and *loss* $\ell : \{0,1\}^2 \to \mathbb{R}$,

Risk:

$$\boxed{R[f] := \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]}$$

where $\mathbb{E}_{\boldsymbol{X}Y}[\cdot]$ is the expectation with respect to $\boldsymbol{X}$ and Y

Optimal decision problem: Given ℓ, find f that minimizes the risk:

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. . . infinite-dimensional problem

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Recall that $f(\boldsymbol{X})$ and Y are binary

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Conditioning on $\boldsymbol{X}$,

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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Conditioning on $\boldsymbol{X}$,

$$\mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big] = \mathbb{E}_{\boldsymbol{X}}\Bigg[\mathbb{E}_Y\Big[\ell\big(f(\boldsymbol{X}),\, Y\big)\,\Big|\,\boldsymbol{X}\Big]\Bigg]$$

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$$\begin{aligned}
\mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big] &= \mathbb{E}_{\boldsymbol{X}}\left[\mathbb{E}_Y\left[\ell\big(f(\boldsymbol{X}),\, Y\big) \,\Big|\, \boldsymbol{X}\right]\right] \\
&= \int_{\mathbb{R}^d} \mathbb{E}_Y\left[\ell\big(f(\boldsymbol{X}),\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right] f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

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If $f(\boldsymbol{x})=0$,

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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If $f(\boldsymbol{x}) = 0$,

$$\mathbb{E}_Y\left[\ell\big(0,\, Y\big)\,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right] = \ell(0,0)\,\mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) + \ell(0,1)\,\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

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If $f(\boldsymbol{x}) = 1$,

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If $f(\boldsymbol{x}) = 1$,

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Optimal decision

Optimal decision

$$f(\boldsymbol{x}) = 0 \qquad \text{if} \qquad \mathbb{E}_Y\left[\ell\big(0,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(Y = 1 \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)}\, \mathbb{P}\big(Y = 0 \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}$$

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Rearranging, the optimal decision is

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Bayes rule

Optimal decision

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$$\frac{f_{\boldsymbol{X}\mid H_1}(\boldsymbol{x} \mid H_1)\, \mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

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Rearranging, the optimal decision is

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$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)\, \mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)\, \mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

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Bayes rule

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$$f(\boldsymbol{x}) = 1 \qquad \text{if} \qquad \frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)} \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}$$

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$$f(\boldsymbol{x}) = 0 \quad \text{if} \quad \mathbb{E}_Y\left[\ell\big(0, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Rearranging, the optimal decision is

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Bayes rule

$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

likelihood ratio test

$$f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)} \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}$$

Optimal decision

$$f(\boldsymbol{x}) = 0 \quad \text{if} \quad \mathbb{E}_Y\left[\ell\big(0, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Bayes rule

$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)\, \mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)\, \mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

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$$f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\underbrace{\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)}}_{\mathcal{L}(\boldsymbol{x})} \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}$$

Optimal decision

$$
\begin{aligned}
f(\boldsymbol{x}) = 0 \quad &\text{if} \quad \mathbb{E}_Y\Big[\ell(0,\,Y)\,\Big|\,\boldsymbol{X}=\boldsymbol{x}\Big] < \mathbb{E}_Y\Big[\ell(1,\,Y)\,\Big|\,\boldsymbol{X}=\boldsymbol{x}\Big] \\
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\end{aligned}
$$

Rearranging, the optimal decision is

$$
f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overset{H_1}{\boxed{Y=1}}\,\big|\,\boldsymbol{X}=\boldsymbol{x}\big) \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)}\,\mathbb{P}\big(\overset{H_0}{\boxed{Y=0}}\,\big|\,\boldsymbol{X}=\boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}
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Bayes rule

$$
\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x}\,|\,H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad\qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x}\,|\,H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}
$$

likelihood ratio test

$$
f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x}\,|\,H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x}\,|\,H_0)} \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}
$$

$\mathcal{L}(\boldsymbol{x})$: likelihood ratio

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \;\; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \;\; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x}) \geq \eta\}}(\boldsymbol{x})$$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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- Indicator function of set $\mathcal{S}$: $\mathbb{1}_{\mathcal{S}}(s) = \begin{cases} 1 & , \text{ if } s \in \mathcal{S} \\ 0 & , \text{ if } s \notin \mathcal{S} \end{cases}$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x}) \geq \eta\}}(\boldsymbol{x})$$

- Indicator function of set $\mathcal{S}$: $\mathbb{1}_{\mathcal{S}}(s) = \begin{cases} 1 & , \text{ if } s \in \mathcal{S} \\ 0 & , \text{ if } s \notin \mathcal{S} \end{cases}$
- *Likelihood ratio:* $\mathcal{L}(\boldsymbol{x}) = \dfrac{f_{\boldsymbol{X}|H_1}\big(\boldsymbol{x} \mid H_1\big)}{f_{\boldsymbol{X}|H_0}\big(\boldsymbol{x} \mid H_0\big)}$

Likelihood ratio test

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The decision that minimizes the risk in a binary hypothesis test is

$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x})\geq\eta\}}(\boldsymbol{x})$$

- Indicator function of set $\mathcal{S}$: $\mathbb{1}_{\mathcal{S}}(s) = \begin{cases} 1 & , \text{ if } s\in\mathcal{S} \\ 0 & , \text{ if } s\notin\mathcal{S} \end{cases}$
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Example in $\mathbb{R}$

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$H_0 \,:\, X = W$

$H_1 \,:\, X = c + W$

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$H_0 \; : \; X = W$

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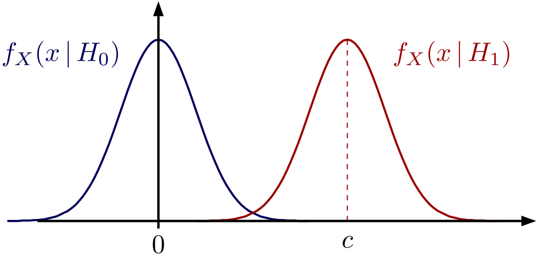

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Compute the decision threshold

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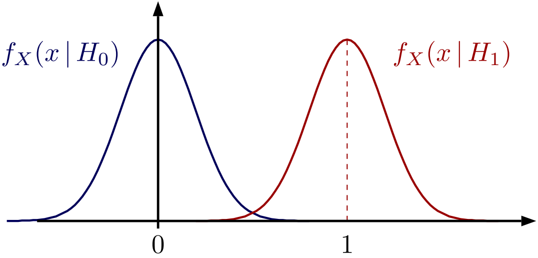

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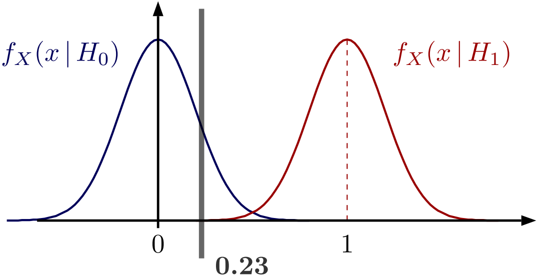

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When class $Y \in \{0,1\}$ is viewed as a parameter of $\mathbb{P}_{\boldsymbol{X}Y}$ to estimate,

- Maximum a posteriori (MAP)
- Maximum likelihood (ML)

can be seen as *likelihood ratio tests*

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$$\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

$$\Longleftrightarrow \qquad f_{\boldsymbol{X}|H_1}(x \,|\, H_1) \cdot \mathbb{P}(H_1) \geq f_{\boldsymbol{X}|H_0}(x \,|\, H_0) \cdot \mathbb{P}(H_0)$$

$$\Longleftrightarrow \qquad \mathcal{L}(\boldsymbol{x}) = \frac{f_{\boldsymbol{X}|H_1}(x \,|\, H_1)}{f_{\boldsymbol{X}|H_0}(x \,|\, H_0)} \geq \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)} = \eta$$

So, select $f(\boldsymbol{x}) = 1$ if $\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$

That is, $$f(\boldsymbol{x}) = \arg\max_i \underbrace{\boxed{\mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big)}}_{\textit{posterior}}$$

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Recall that MAP rule minimizes probability of incorrect decision:

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Recall that MAP rule minimizes probability of incorrect decision:

$$\mathbb{P}\big(\text{error}\big) = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,,\, H_0\big) + \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,,\, H_1\big)$$

Maximum Likelihood (ML)

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$$f(\boldsymbol{x}) \;=\; \arg\max_i \;\; \mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

$$=\; \arg\max_i \;\; \frac{f_{\boldsymbol{X}|Y}(\boldsymbol{x} \,|\, Y = i) \cdot \mathbb{P}(Y = i)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \Big(\text{Bayes rule}\Big)$$

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The optimal decision (MAP) is

$$
\begin{aligned}
f(\boldsymbol{x}) &= \arg\max_i \ \mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\
&= \arg\max_i \ \frac{f_{\boldsymbol{X}|Y}(\boldsymbol{x} \,|\, Y = i) \cdot \mathbb{P}(Y = i)}{f_{\boldsymbol{X}}(\boldsymbol{x})} && \Big(\text{Bayes rule}\Big) \\
&= \arg\max_i \ f_{\boldsymbol{X}|Y}\Big(\boldsymbol{x} \,\Big|\, Y = i\Big) && \Big(\mathbb{P}(Y = i) = \frac{1}{2}\Big)
\end{aligned}
$$

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&=\; \underbrace{\boxed{\arg\max_i \;\; f_{\boldsymbol{X}|Y}\Big(\boldsymbol{x} \,\Big|\, Y = i\Big)}}_{\textit{maximum likelihood}} && \Big(\mathbb{P}(Y = i) = \frac{1}{2}\Big)
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\end{aligned}
$$

This corresponds to a likelihood ratio test with $\eta = 1$

Types of errors and successes

Types of errors and successes

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
H_1 is true		

Types of errors and successes

True Positive Rate (TPR)

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
H_1 is true		

Types of errors and successes

True Positive Rate (TPR)
power, sensitivity, recall

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$$\text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$$

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H_0 is true		
H_1 is true		TPR

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True Positive Rate (TPR)
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$$\text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$$

False Positive Rate (FPR)
type I error, false alarm

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
H_1 is true		TPR

Types of errors and successes

True Positive Rate (TPR) $\qquad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

power, sensitivity, recall

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type I error, false alarm

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
H_1 is true		TPR

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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

power, sensitivity, recall

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type I error, false alarm

True Negative Rate (TNR)

specificity

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
H_1 is true		TPR

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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

power, sensitivity, recall

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True Negative Rate (TNR)

specificity

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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True Negative Rate (TNR) $\quad \text{TNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_0\big)$

specificity

False Negative Rate (FNR)

type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
H_1 is true		TPR

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True Positive Rate (TPR) $\qquad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

power, sensitivity, recall

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type I error, false alarm

True Negative Rate (TNR) $\qquad \text{TNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_0\big)$

specificity

False Negative Rate (FNR)

type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
H_1 is true	FNR	TPR

Types of errors and successes

True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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type I error, false alarm

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False Negative Rate (FNR) $\quad \text{FNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_1\big)$

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H_0 is true	TNR	FPR
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False Negative Rate (FNR) $\qquad \text{FNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_1\big)$

type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
H_1 is true	FNR	TPR

Types of errors and successes

True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

power, sensitivity, recall

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type I error, false alarm

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specificity

False Negative Rate (FNR) $\quad \text{FNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_1\big)$

type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
H_1 is true	FNR $\boldsymbol{\beta}$	TPR

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Precision: $\mathbb{P}\big(H_1 \,\big|\, f(\boldsymbol{X}) = 1\big)$

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Precision: $$\mathbb{P}\big(H_1 \,\big|\, f(\boldsymbol{X}) = 1\big) = \frac{\mathsf{TPR} \cdot \mathbb{P}(H_1)}{\mathsf{TPR} \cdot \mathbb{P}(H_1) + \mathsf{FPR} \cdot \mathbb{P}(H_0)}$$

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$\boldsymbol{F_1}$**-score:** harmonic mean between precision and recall (TPR):

$$F_1 \;=\; \frac{2}{\frac{1}{\mathbb{P}(H_1 \mid f(\boldsymbol{X})=1)} + \frac{1}{\mathsf{TPR}}}$$

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
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Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
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α and β are in *conflict:*

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
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α and β are in *conflict:*

- $\alpha \downarrow \implies \beta \uparrow$: and vice-versa

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True hypothesis	$f(\boldsymbol{X})=0$	$f(\boldsymbol{X})=1$
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- Both can decrease only by observing more data

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It turns out that likelihood ratio tests are *Pareto optimal*

Neyman-Pearson Lemma

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Theorem

Let $f(\boldsymbol{x})$ be a decision rule (det. or prob.)

Neyman-Pearson Lemma

Theorem

Let $f(\boldsymbol{x})$ be a decision rule (det. or prob.) with FPR and FNR

$$\alpha = \mathbb{P}\Big(f(\boldsymbol{x}) = 1 \,\Big|\, H_0\Big) \qquad\qquad \beta = \mathbb{P}\Big(f(\boldsymbol{x}) = 0 \,\Big|\, H_1\Big)$$

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$$\alpha_{\mathsf{LRT}} = \mathbb{P}\Big(f_{\mathsf{LRT}}(\boldsymbol{x}) = 1 \,\Big|\, H_0\Big) \qquad \beta_{\mathsf{LRT}} = \mathbb{P}\Big(f_{\mathsf{LRT}}(\boldsymbol{x}) = 0 \,\Big|\, H_1\Big)$$

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$$\beta \le \beta_{\mathsf{LRT}} \quad \Longrightarrow \quad \alpha \ge \alpha_{\mathsf{LRT}}$$

And the same relations hold with strict inequalities ($<, >$)

Proof

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$$\mathbb{P}\Big(\mathsf{error}_{\mathsf{MAP}}\Big)$$

$$= \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\, H_0\Big) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 0,\, H_1\Big)$$

Proof

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$$\begin{aligned} &\mathbb{P}\Big(\mathsf{error}_{\mathsf{MAP}}\Big) \\ &= \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\ H_0\Big) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 0,\ H_1\Big) \\ &= \ \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1 \,\Big|\, H_0\Big)\ \mathbb{P}(H_0) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 0 \,\Big|\, H_1\Big)\ \mathbb{P}(H_1) \end{aligned}$$

Proof

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&\mathbb{P}\Big(\text{error}_{\mathsf{MAP}}\Big) \\
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That is, $\big(\alpha - \alpha_{\mathsf{MAP}}\big)\, \mathbb{P}(H_0) + \big(\beta - \beta_{\mathsf{MAP}}\big)\, \mathbb{P}(H_1) \ \ge \ 0$

Proof

- $\left(\alpha - \alpha_{\mathsf{MAP}}\right) \mathbb{P}(H_0) + \left(\beta - \beta_{\mathsf{MAP}}\right) \mathbb{P}(H_1) \;\geq\; 0$

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$\square$

Receiver Operating Characteristic (ROC)

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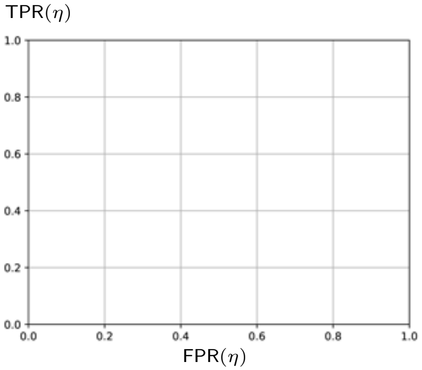

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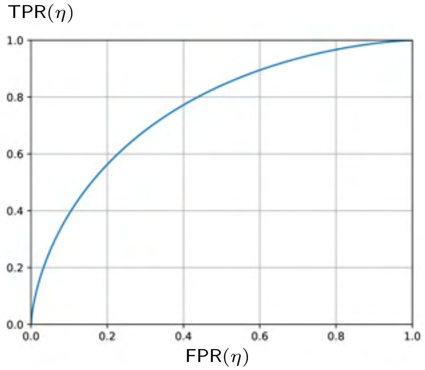

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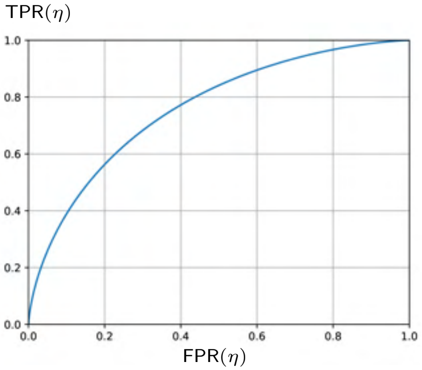

ROC curve

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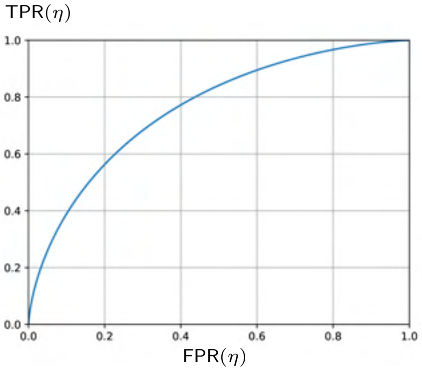

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Neyman-Pearson implies *Pareto optimality*

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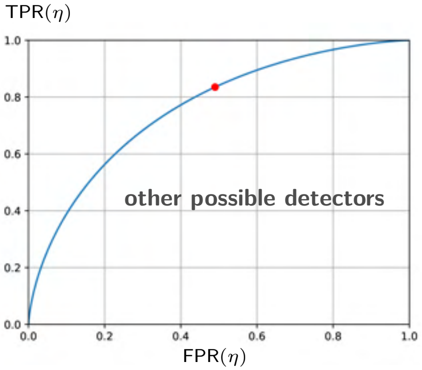

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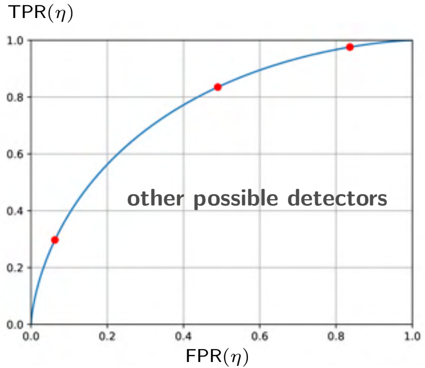

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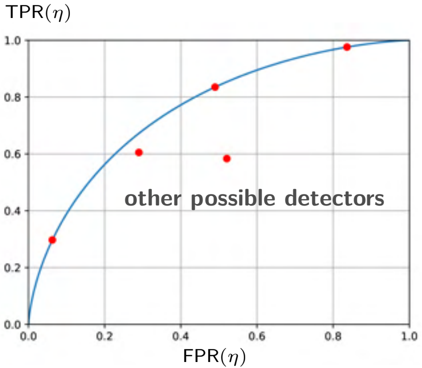

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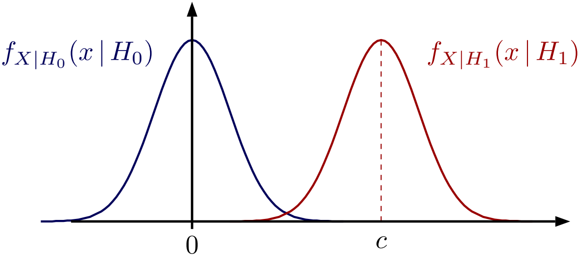
Example

Example



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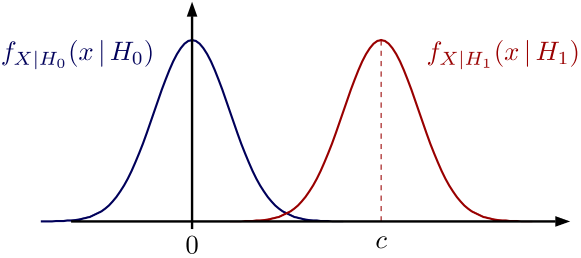

$$f_{X|H_0}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

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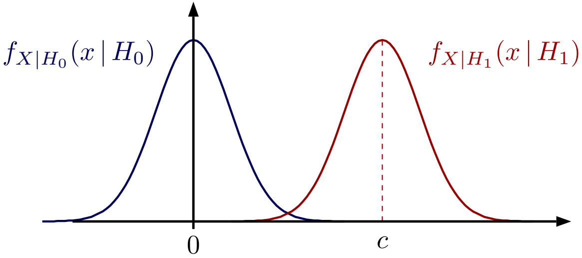

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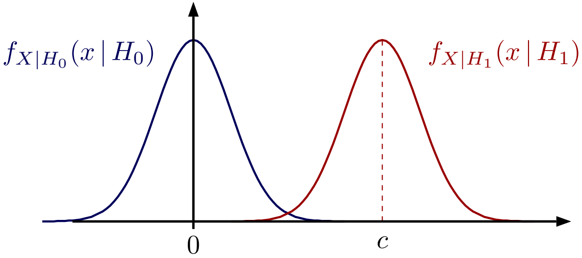

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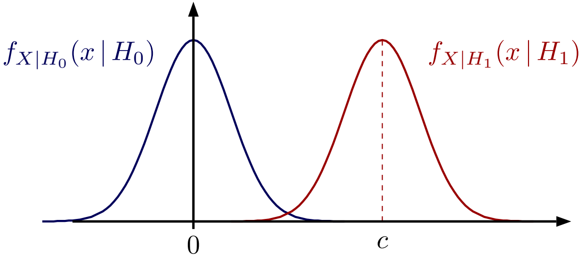
Likelihood ratio test:

Example



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$$\mathcal{L}(x) = \frac{f_{X|H_1}(x)}{f_{X|H_0}(x)} \geq \eta$$

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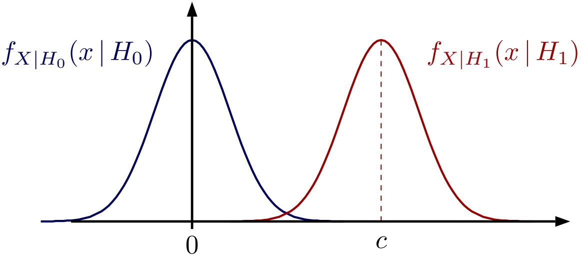

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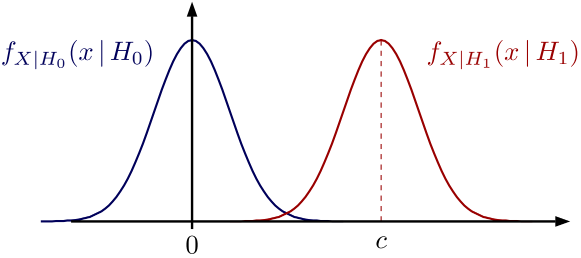

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$\mathsf{TPR}(\eta)$

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$$\mathsf{TPR}(\eta) = \mathbb{P}\Big(X \geq \gamma \,\Big|\, H_1 \Big)$$

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Example

$$x \geq \frac{c}{2} + \frac{\sigma^2}{c} \log \eta := \gamma$$

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Similarly,

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ROC curve for different values of SNR

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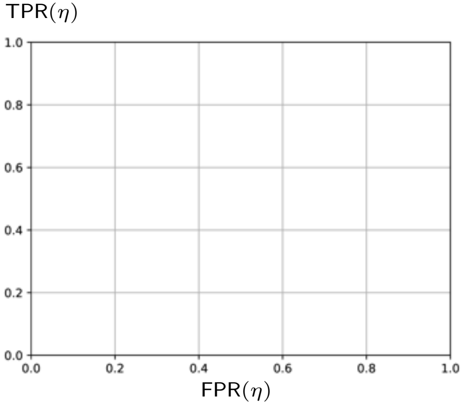

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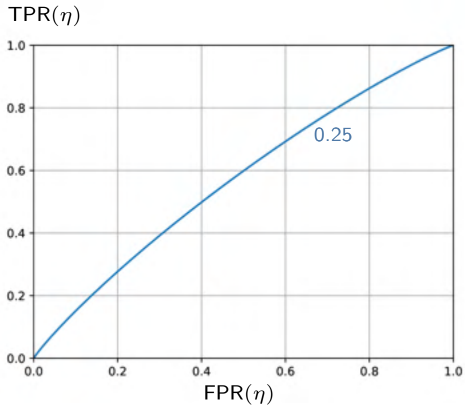

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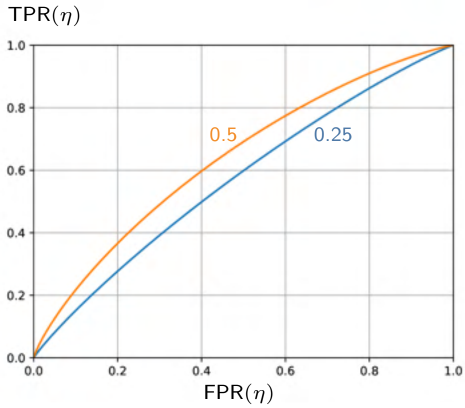

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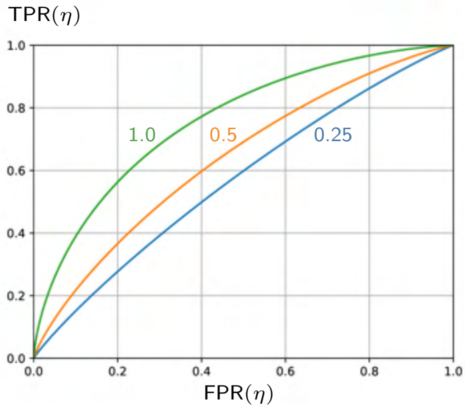

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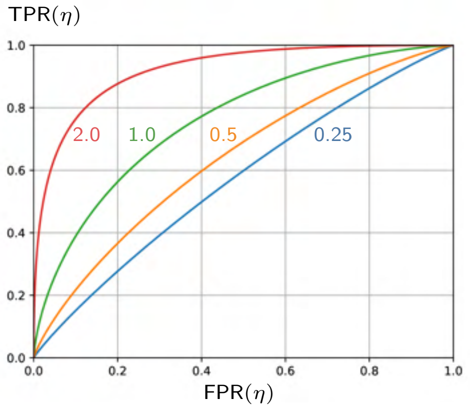

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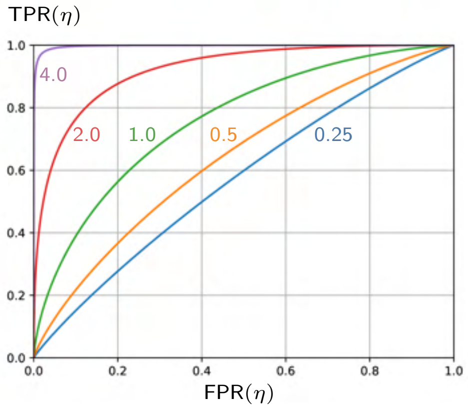

Properties of ROC curves

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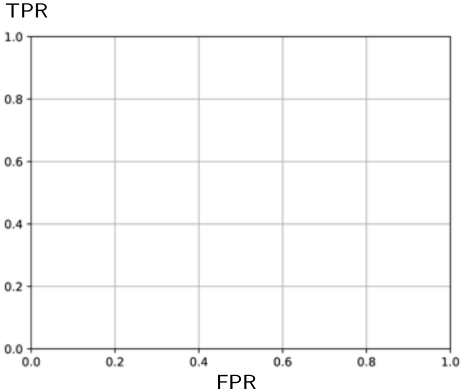

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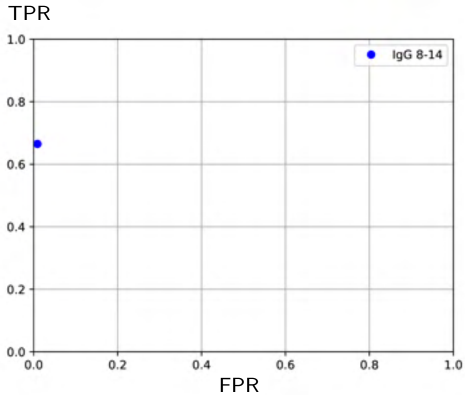

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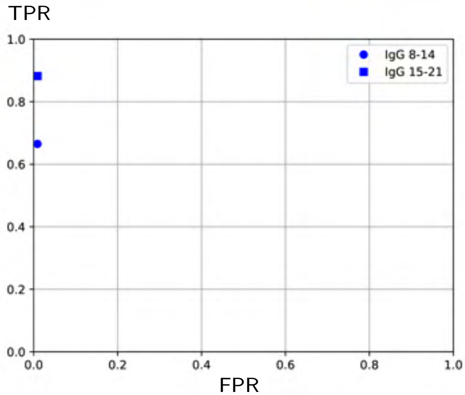

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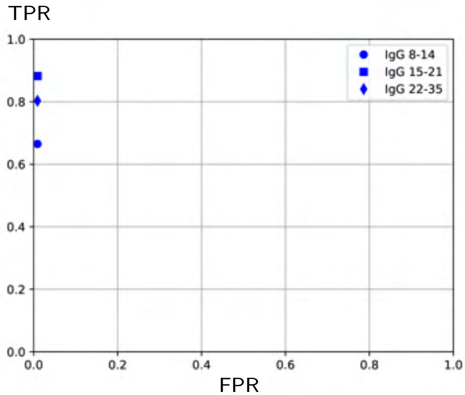

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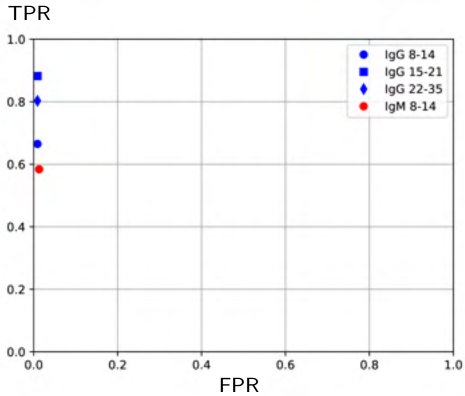

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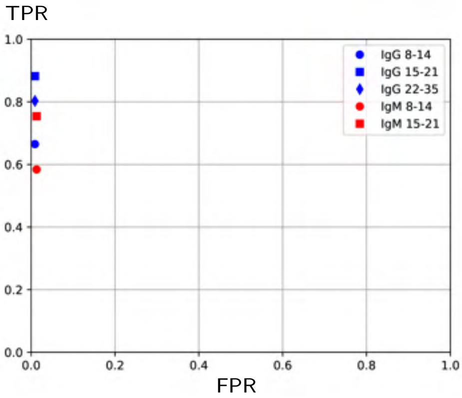

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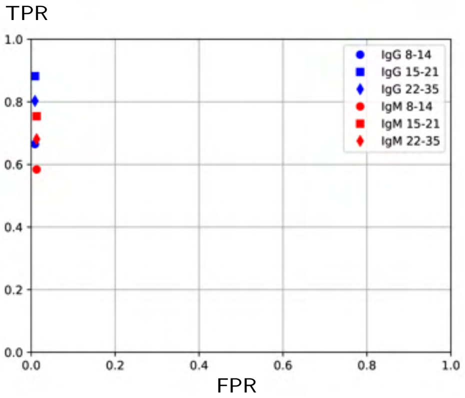

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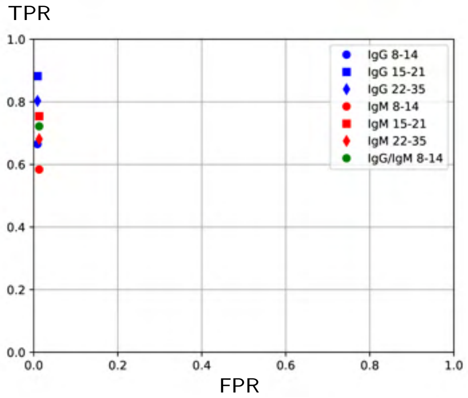

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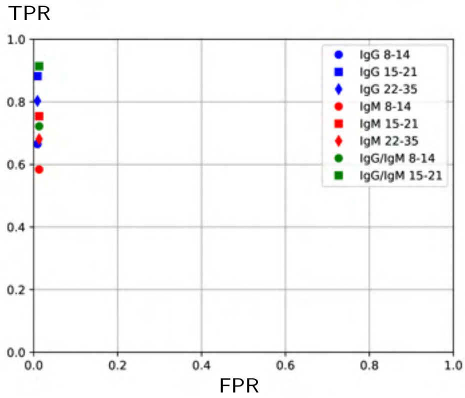

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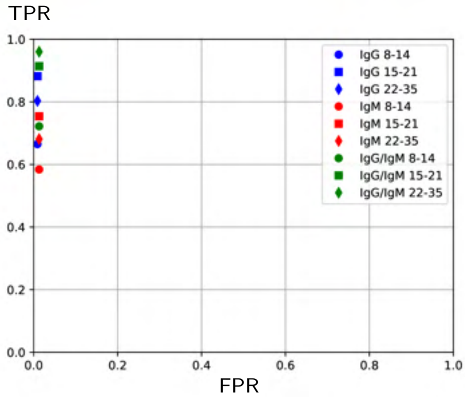

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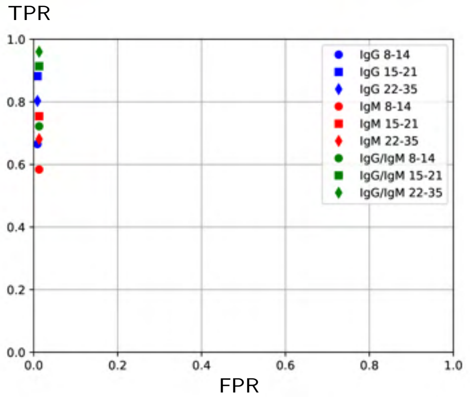

Deeks et al, **Antibody tests for identification of current and past infection with SARSCoV2**, Cochrane Database of Systematic Reviews, Issue 6, 2020

Looking Ahead: Empirical Risk Minimization

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Conclusions

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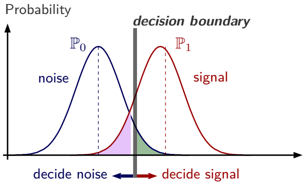

Optimal decision problems (binary case)

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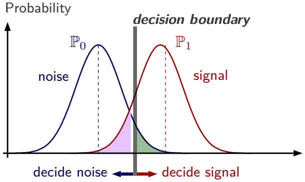

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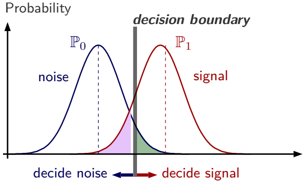

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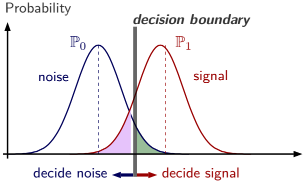

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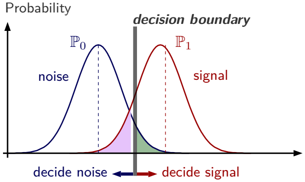

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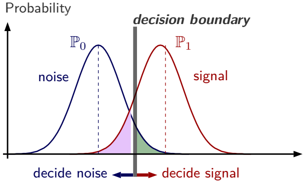

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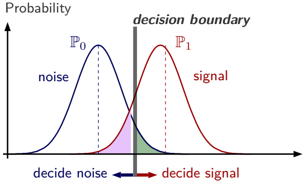

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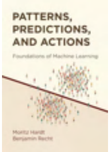

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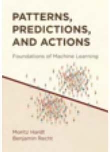
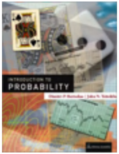
Princeton University Press, Oct, 2022

References





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Patterns, Predictions, and Actions

Princeton University Press, Oct, 2022

D. P. Bertsekas, J. N. Tsitsiklis

Introduction to Probability

Athena Scientific, 2nd edition, 2008

T. Fauwcett, "An introduction to ROC analysis," Pattern Recognition Letters,

Vol. 27, pp. 861-874, 2006.

H. L. Van Trees, **Detection, Estimation, and Modulation: Part I**,

John Wiley & Sons, 2001.