# Source Separation and Beamforming Background

Ian K. Proudler

Dept. of Electronic & Electrical Engineering, Univerity of Strathclyde

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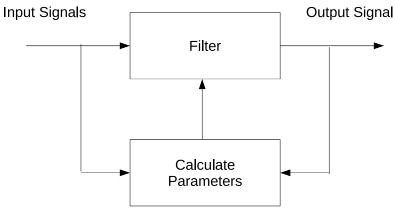
# Source Separation and Beamforming Background: Overview

1. Overview
2. Signal Separation
3. Non-adaptive beamforming
4. Adaptive signal processing for beamforming
5. Application of linear algebra to array problems
6. More adaptive signal processing for beamforming
7. Blind source separation
8. Summary

# Signal Separation

- Signal separation requires two components:




- A parametrised mechanism to separate the signals (a "filter")
- A means to select the parameters
- Performance limited by 'optimum' filter

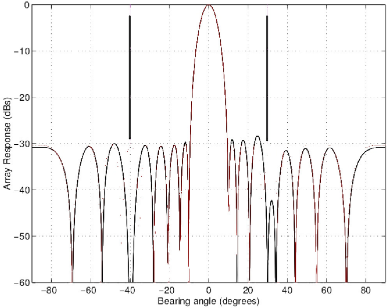
- Conventionally we have two "filter" mechanisms:
  - Temporal filter – separate by frequency
  - Spatial filter (aka beamformer) – separate by AOA
- Could use a nonlinear filter (if you can think of one!)
- We will focus on narrowband beamforming in this talk
- Broadband beamforming requires a space-time filter

# Signal Separation

- Performance limited by 'optimum' filter
- Narrowband beamforming → signals must have sufficiently different angles of arrival (AOA)
- Parameter selection – the interesting part
- Three cases:
  - Non-adaptive – we know everything about the scenario
  - "Adaptive" – we don't know everything
  - "Blind" – we don't know anything (sort of)
- Important parameters:
  - AOA of signals
  - Array calibration
  - Noise statistics

# Non-Adaptive Source Separation

- Covered in talk by Prof. Weiss




- Beamformer weights via constrained optimisation (offline)
- Gain towards wanted signal = 1
- Gain towards other signals = 0
- Noise gain as small as possible

- Lots of good optimisation algorithms
  (DSP text books e.g. Rabiner & Gold - Temporal filters but basically the same for beamforming)
- Only $(N-1)$ nulls
- Spatially distributed noise can't be removed only suppressed

# Adaptive Source Separation

- Usually called Adaptive Beamforming
- Basic theory introduced in talk by Prof. Weiss
- Assume the known parameters are:
  - AOA of the wanted signal(s)
  - Array calibration
- Beamformer weights via constrained optimisation but online this time
- Gain towards wanted signal $= 1$
- Minimise energy of output
- NB. Could use an AOA algorithm here and fixed beamforming but computationally costly

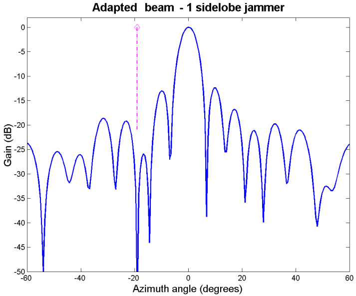
# Adaptive Source Separation

- Nomenclature:
  - Steering vectors $\mathbf{A}$ relate array data to signals (see Prof. Weiss' talk)
$$\mathbf{X} = \mathbf{AS}$$
  - Beamformer weights: $\mathbf{w}$
  - Sensor data at time $n$: $\mathbf{x}(n)$
  - Output at time $n$: $y(n) = \mathbf{w}^H \mathbf{x}(n)$
  - Energy in output: $J = \sum_{n=0}^{N-1} |y(n)|^2 = ||\mathbf{w}^H \mathbf{X}\mathbf{X}^H \mathbf{w}||_2^2$
  - Data matrix: $\mathbf{X} = [\mathbf{x}(0), \mathbf{x}(1), ..., \mathbf{x}(N-1)]$
  - Constraint: $\mathbf{w}^H \mathbf{a}(\theta) = 1$
  - Sample covariance matrix: $\mathbf{R} = \mathbf{X}\mathbf{X}^H$

# Minimum Variance Distortionless Response (MVDR)

- Minimum Variance := Minimise energy of output
- Distortionless Response := Gain towards wanted signal = 1

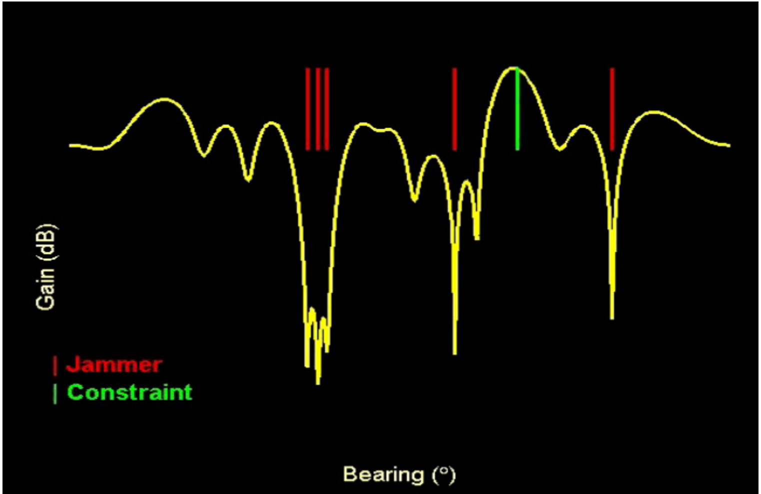
$$\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}$$




- Gain towards wanted signal = 1
- Small gain (null) towards other signal
- Noise gain not controlled
  In fact adapted to that particular noise realization

# Minimum Variance Distortionless Response (MVDR)

- Multiple noise realizations (blocks of data)



# Minimum Variance Distortionless Response (MVDR)

- Stabilisation procedures: there are many different ways of reducing the effects of adapting to the noise realizations.
- All effectively try to 'remove' influence of noise
- Diagonal loading

$$\mathbf{w} = \text{Arg Min}\left(||\mathbf{w}^H\left(\mathbf{R}+\mu I\right)\mathbf{w}||_2^2\right) st.\mathbf{w}^H\mathbf{a}(\theta) = 1$$

- Penalty Function Method

$$\mathbf{w} = \text{Arg Min}\left(||\mathbf{w}^H\mathbf{R}\mathbf{w}||_2^2 + \kappa||\mathbf{w}-\mathbf{w}_0||_2^2\right)$$

  "Soft" constraint makes the adapted beam pattern lie close to the desired pattern.
- "Noise" subspace manipulation: Average noise subspace eigenvalues – need some Linear Algebra

# Linear Algebra

- MVDR weight vector depends on covariance matrix $\mathbf{R}$
- This matrix has structure that can be exploited
- We can use linear algebra to study / manipulate the covariance matrix

- Topics:
  - Eigenvalue decomposition of Hermitian matrix
  - Rotation Matrices
  - Eigenvectors are not steering vectors
  - Eigenvalue spectrum
  - Signal and Noise Subspaces
  - Singular Value Decomposition

# Eigenvalue Decomposition

- The covariance matrix is Hermitian (symmetric)

$$\mathbf{R}^H = \left(\mathbf{X}\mathbf{X}^H\right)^H = \mathbf{X}\mathbf{X}^H = \mathbf{R}$$

- Eigenvalue decomposition of Hermitian matrix

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$$

- Eigenvectors: $\mathbf{U}$ is a unitary matrix (a rotation in N-D space)

$$\mathbf{U}^H\mathbf{U} = I$$

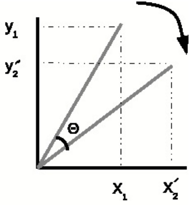
- Eigenvalues: $\mathbf{\Lambda}$ is diagonal, all elements are real and $\geq 0$
- Rank of $\mathbf{R}$ is number of non-zero eigenvalues

# Calculating $\mathbf{U}$: Rotation Matrices

- Eigenvectors: $\mathbf{U}$ is a unitary matrix $\mathbf{U}^H\mathbf{U} = I$
- Hence length preserving:

$$||\mathbf{U}\mathbf{x}|| = \mathbf{x}^H\mathbf{U}^H\mathbf{U}\mathbf{x} = \mathbf{x}^H\mathbf{x} = ||\mathbf{x}||$$

- A Rotation in N-D space
- Often calculated by multiple applications of 2-D (Givens) rotations:




$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta)^* & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2' \\ y_2' \end{bmatrix}$$

$(y_2' = 0)$

# Calculating $\mathbf{U}$: Rotation Matrices

- In N-D space $\mathbf{U}$ can be build up by embedding Givens rotation in a unit matrix

$$\mathbf{U} = \ldots \begin{bmatrix} \cos(\theta_{i+1}) & 0 & 0 & 0 & \sin(\theta_{i+1}) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\sin(\theta^*_{i+1} & 0 & 0 & 0 & \cos(\theta_{i+1}) \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta_i) & 0 & \sin(\theta_i) & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -\sin(\theta_i)^* & 0 & \cos(\theta_i) & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \ldots$$

- For the EVD: $\mathbf{U}^H\mathbf{R}\mathbf{U} = \mathbf{\Lambda}$ is diagonal
- So want to set off-diagonal elements to zero i.e. choose $y'_2 = 0$.
- N-D rotations will be useful for blind signal separation theory

# Eigenvalue Decomposition

- Contrary to common belief the eigenvectors are not steering vectors
- Data and Covariance Matrices

$$\mathbf{X} = \mathbf{A}\mathbf{S} \qquad \mathbf{R} = \mathbf{X}\mathbf{X}^H = \mathbf{A}\mathbf{S}\mathbf{S}^H\mathbf{A}^H$$

- For independent signals $\mathbf{D} \equiv \mathbf{S}\mathbf{S}^H$ is diagonal with $\geq 0$ entries.
- Consider the eigenvalue Decomposition

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H = \mathbf{A}\mathbf{D}\mathbf{A}^H$$

- Tempting to assume that

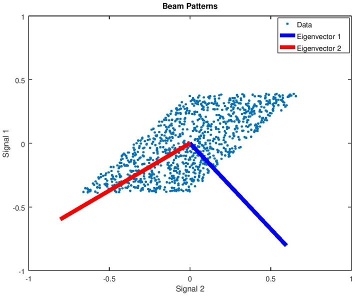
$$\mathbf{U}\mathbf{\Lambda}^{1/2} = \mathbf{A}\mathbf{D}^{1/2}$$

  which would mean that the eigenvectors are proportional to the steering vectors
- But there is an implied 'hidden' unitary matrix (SVD)

$$\mathbf{X} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^H \Rightarrow \mathbf{R} = \mathbf{U}\mathbf{\Lambda}^{1/2}\cancel{\mathbf{V}^H\mathbf{V}}\mathbf{\Lambda}^{1/2}\mathbf{U}^H$$

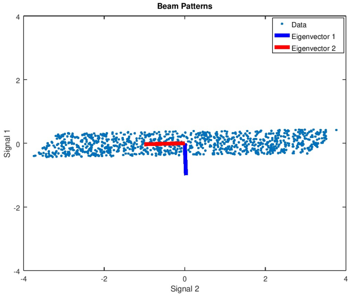
# Eigenvalue Decomposition

- But sometimes eigenvectors are nearly steering vectors
- Eigenvectors of covariance matrix point in direction of maximum energy whilst being orthogonal to each other
- Eigenvectors approximately steering vectors when powers are dissimilar
- Scatter plots:




2

equal power signals




2

signals with power ratio 10:1

# Eigenvalue Decomposition

- The EVD can, however, separate 'signals' from 'noise' if SNR is high enough. Consider two signals

$$\mathbf{X} = \mathbf{a}(\theta_1)\mathbf{s}_1^T + \mathbf{a}(\theta_2)\mathbf{s}_2^T + \mathcal{N}$$

- Covariance matrix

$$\mathbf{R} = \mathbf{X}\mathbf{X}^H = \mathbf{A}\mathbf{D}\mathbf{A}^H + \sigma^2 I$$

- Only two signals so $\mathbf{A}\mathbf{D}\mathbf{A}^H$ is rank two. EVD:

$$\mathbf{A}\mathbf{D}\mathbf{A}^H = \mathbf{U}\begin{bmatrix} \mathbf{\Lambda}_\mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}\mathbf{U}^H$$

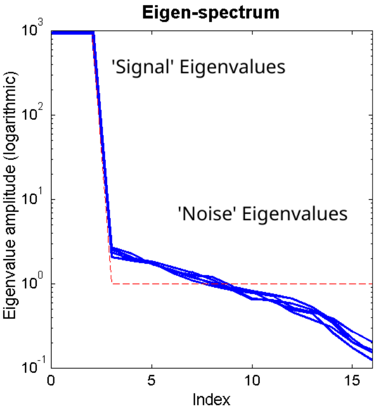
- Covariance matrix EVD (add noise)

$$\mathbf{R} = \mathbf{U}\begin{bmatrix} \mathbf{\Lambda}_\mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}\mathbf{U}^H + \sigma^2 I = \mathbf{U}\begin{bmatrix} \mathbf{\Lambda}_\mathbf{A} + \sigma^2 I & 0 \\ 0 & \sigma^2 I \end{bmatrix}\mathbf{U}^H$$

# Eigenvalue Spectrum

- Eigenvalue spectrum

$$\begin{bmatrix} \mathbf{\Lambda}_{\mathbf{A}} + \sigma^2 I \\ \sigma^2 I \end{bmatrix}$$




- Two large eigenvalues
- Five noise realizations
- Noise eigenvalues not the same and not equal what theory suggests – finite data

# Signal and Noise Subspaces

- Consider the covariance matrix EVD (replace 'theoretical' $\sigma^2 I$ by $\mathbf{\Lambda_N}$)

$$\mathbf{R} = \mathbf{U} \begin{bmatrix} \mathbf{\Lambda_A} + \mathbf{\Lambda_{N1}} & 0 \\ 0 & \mathbf{\Lambda_{N2}} \end{bmatrix} \mathbf{U}^H$$

- Partition eigenvectors (assuming $\mathbf{\Lambda_A} + \mathbf{\Lambda_{N1}} > \mathbf{\Lambda_{N2}}$)

$$\mathbf{U} = \begin{bmatrix} \mathbf{U_1} & \mathbf{U_2} \end{bmatrix}$$

- Orthogonal subspaces: 'Signal and Noise' and 'Noise'

$$\mathbf{U_1}^H \mathbf{U_1} = I \qquad \mathbf{U_2}^H \mathbf{U_2} = I \qquad \mathbf{U_1}^H \mathbf{U_2} = 0$$

- Then the covariance matrix EVD becomes

$$\mathbf{R} = \underbrace{\mathbf{U_1}\left(\mathbf{\Lambda_A} + \mathbf{\Lambda_{N1}}\right)\mathbf{U_1}^H}_{\text{Signal plus Noise}} + \underbrace{\mathbf{U_2}\left(\mathbf{\Lambda_{N2}}\right)\mathbf{U_2}^H}_{\text{Noise only}}$$

- Will use this later to remove some of the noise.

# Singular Value Decomposition

- Not all matrices of interest are Hermitian
- Singular value decomposition of a matrix $\mathbf{X}$:

$$\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

  If $\mathbf{X}$ is $N \times M$: $\mathbf{U}$ is $N \times N$, $\mathbf{\Sigma}$ is $N \times M$, and $\mathbf{V}$ is $M \times M$
- Singular vectors: $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices
- Singular values: $\mathbf{\Sigma}$ is diagonal, all elements are $\geq 0$
- Rank of $\mathbf{X}$ is number of non-zero singular values
- Relation to EVD

$$\mathbf{R} = \mathbf{X}\mathbf{X}^H = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{V}\mathbf{\Sigma}\mathbf{U}^H = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^H$$

  Eigenvalues are the square of the singular values
- Recall we used the SVD to show that eigenvalues are not steering vectors

# Stabilized MVDR Beamformer

- Back to beamforming ....
- Recall basic MVDR beamformer suffers from weight jitter because of the noise
- Covariance matrix EVD

$$\mathbf{R} = \begin{bmatrix} \mathbf{U_1} & \mathbf{U_2} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda_A} + \mathbf{\Lambda}_{N1} & 0 \\ 0 & \mathbf{\Lambda}_{N2} \end{bmatrix} \begin{bmatrix} \mathbf{U_1}^H \\ \mathbf{U_2}^H \end{bmatrix}$$

$$\mathbf{R} = \underbrace{\mathbf{U_1} \left( \mathbf{\Lambda_A} + \mathbf{\Lambda}_{N1} \right) \mathbf{U_1}^H}_{\text{Signal plus Noise}} + \underbrace{\mathbf{U_2} \left( \mathbf{\Lambda}_{N2} \right) \mathbf{U_2}^H}_{\text{Noise only}}$$

- Idea 1: Subspace Projection: remove $\mathbf{\Lambda}_{N2}$ using $\mathbf{U_1}^H \mathbf{U_2} = 0$
- Project data onto signal plus noise subspace

$$\tilde{\mathbf{X}} = \overbrace{\mathbf{U_1} \mathbf{U_1}^H}^{\text{Projection Operator}} \mathbf{X}$$

# Signal and Noise Subspaces

- Orthogonal subspaces: $\mathbf{U_1}^H\mathbf{U_2} = 0$

$$\tilde{\mathbf{R}} = \tilde{\mathbf{X}}\tilde{\mathbf{X}}^H = \mathbf{U_1}\mathbf{U_1}^H\mathbf{X}\mathbf{X}^H\mathbf{U_1}^H\mathbf{U_1} = \mathbf{U_1}\mathbf{U_1}^H\mathbf{R}\mathbf{U_1}^H\mathbf{U_1}$$
$$= \begin{bmatrix} \mathbf{U_1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda_A} + \mathbf{\Lambda}_{N1} & 0 \\ 0 & \mathbf{\Lambda}_{N2} \end{bmatrix} \begin{bmatrix} \mathbf{U_1}^H \\ 0 \end{bmatrix}$$
$$= \mathbf{U_1}\left[\mathbf{\Lambda_A} + \mathbf{\Lambda_{N1}}\right]\mathbf{U_1}^H \equiv \mathbf{U} \begin{bmatrix} \mathbf{\Lambda_A} + \mathbf{\Lambda_{N1}} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{U}^H$$

- Thus $\tilde{\mathbf{R}}$ is a covariance matrix for a "noise free" case but the signals are noisy
- Issues with rank deficient $\tilde{\mathbf{R}}$ since it is not invertible

# Stabilized MVDR Beamformer

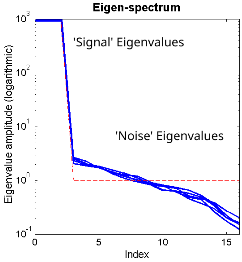
- Idea 2: Average noise eigenvalues
- Project data onto noise subspace to estimate noise power $\sigma^2$

$$\mathbf{N} = \mathbf{U_2}\mathbf{U_2}^H\mathbf{X}$$

- Calculate a $\sigma^2$ over several snapshots

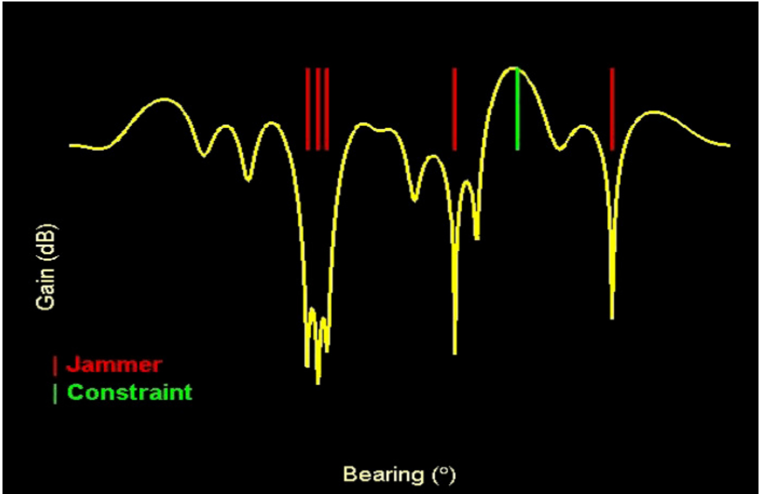
$$\tilde{\mathbf{R}} = \mathbf{U}\begin{bmatrix} \mathbf{\Lambda_A} + \Lambda_{N1} & 0 \\ 0 & \sigma^2 I \end{bmatrix}\mathbf{U}^H$$

- Need to decide how to partition $U$ into $U_1$ and $U_2$.
- Look at eigenvalues
- Can use simple thresholding or more complicated information theory.



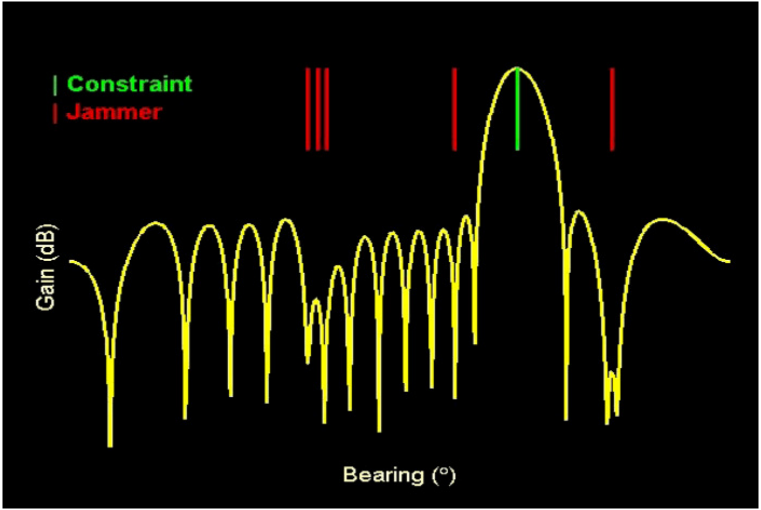
# Minimum Variance Distortionless Response (MVDR)

- Recall multiple noise realizations (blocks of data) caused jitter



# Stabilized MVDR Beamformer

- Average noise eigenvalues



# Array Calibration Errors

- MVDR minimises power in output signal.
- $\mathbf{w} = 0$ would do this but also removes wanted signal
- 'Look direction' constraint protects the wanted signal

$$\mathbf{w}^H \mathbf{a}(\theta) = 1$$

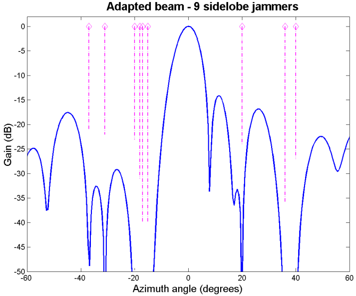
- What if $\mathbf{a}(\theta)$ is incorrect?
- Wanted signal looks like an unwanted one!
- Add extra constraints
  - More that one 'Look direction' constraint
  - Flatten main lobe – gradient constraint
- Incorporate calibration into problem and solve ...

# Linearly Constrained Minimum Variance (LCMV)

- MVDR has only one constraint; can we have more?
- LCMV algorithm
  - Minimum Variance = Minimise energy of output
  - Linearly Constrained = More than one constraint (e.g. could have fixed null)

$$\mathbf{w}^H \mathbf{C} = \mathbf{g}^T$$

- Solution: $$\mathbf{w} = \mathbf{R}^{-1}\mathbf{C}\left(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C}\right)^{-1}\mathbf{g}$$




- Gain in wanted direction = 1
- Gain towards other known directions = 0
- Could also adapt to unknown signals like the MVDR

# Linearly Constrained Minimum Variance (LCMV)

- LCMV is a constrained minimisation problem

$$\mathbf{w} = \text{Arg Min}\left(||\mathbf{w}^H\mathbf{R}\mathbf{w}||_2^2\right) st.\mathbf{w}^H\mathbf{C} = \mathbf{g}^T$$

- If there are $M$ constraints, $M$ components of $\mathbf{w}$ are effectively fixed
- Thus only $N - M$ 'degrees of freedom' in the choice of $\mathbf{w}$
  i.e. can only null out $N - M$ unknown signals
- Thus have to have $N - M > 0$
- Sometimes the constraints can be linearly dependent or nearly so .....

# Linearly Constrained Minimum Variance (LCMV)

- Consider

$$\mathbf{w}^H \mathbf{C} = \mathbf{g}^T$$

- or

$$\left[\mathbf{w}^H \mathbf{C} - \mathbf{g}^T\right] = \left[\mathbf{w}^H, -1\right] \begin{bmatrix} \mathbf{C} \\ \mathbf{g}^T \end{bmatrix} = 0$$

- Take SVD

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{g}^T \end{bmatrix} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

- Note $V$ is full rank so we have

$$\left[\mathbf{w}^H, -1\right] \mathbf{U}\mathbf{\Sigma} = 0$$

# Linearly Constrained Minimum Variance (LCMV)

- Say $(M - R)$ singular values $(\mathbf{\Sigma_2})$ are small

$$\mathbf{U\Sigma} = \begin{bmatrix} \mathbf{U_1} & \mathbf{U_2} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma_1} & 0 \\ 0 & \mathbf{\Sigma_2} \end{bmatrix}$$
$$\approx \begin{bmatrix} \mathbf{U_1} & \mathbf{U_2} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{U_1\Sigma_1} & 0 \end{bmatrix}$$

- Then an approximation to our problem is

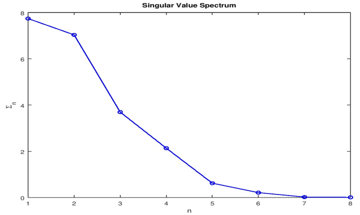
$$\left[\mathbf{w}^H, -1\right] \mathbf{U_1\Sigma_1} = 0$$

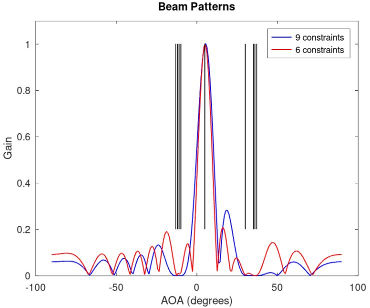
- Alternatively, writing $\mathbf{U_1\Sigma_1} = \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{g}}^T \end{bmatrix}$ the approximate problems is

$$\mathbf{w}^H \tilde{\mathbf{C}} = \tilde{\mathbf{g}}^T$$

- Note $\tilde{\mathbf{C}}$ only has $R < M$ columns

# Linearly Constrained Minimum Variance (LCMV)

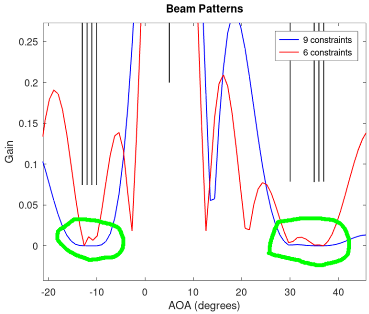



- Constraint matrix singular value spectrum
- 3 small singular values
- 6 constraints nearly as good as 9 constraints?

- Beam patterns
- 16 sensors
- blue - 9 Constraints
- red - 6 Constraints
- Beam patterns similar at constraint points

# Linearly Constrained Minimum Variance (LCMV)




- Beam patterns
- blue - 9 Constraints
- red - 6 Constraints

- Constraints not strictly achieved due to non-zero singular values
- Threshold on singular values should be set by acceptable ‘null’ gain

# Blind Source Separation

- For adaptive beamforming, we assumed we know:
  - AOA of the wanted signal(s)
  - Array calibration
- What if we don't know this information?
  $\Rightarrow$ Blind Source Separation or Independent Component Analysis
- Assume
  - The source signals are statistically independent
  - No more than one Gaussian signal
    (higher order moments of Gaussian signal are zero)
  - Not interested in absolute amplitude of the signals
- Recall that

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathcal{N}$$

- Define SVD of A

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$$

- If we can calculate $\mathbf{U}$, $\boldsymbol{\Sigma}$ and $\mathbf{V}$ we can calculate $\mathbf{A}$
- If we know $\mathbf{A}$ we can invert it and recover $\mathbf{S} \approx \mathbf{A}^{-1}\mathbf{X}$

# Blind Source Separation

- The covariance matrix is

$$\mathbf{R} = \mathbf{X}\mathbf{X}^H = \mathbf{A}\mathbf{D}\mathbf{A}^H + \sigma^2 I$$

- Assume that the source signals are statistically independent and have unit power i.e. $\mathbf{D} = I$.
- (If not, redefine array manifold $\mathbf{A}$ so that $\mathbf{A} \leftarrow \mathbf{A}\mathbf{D}^{\frac{1}{2}}$)
- So the covariance matrix is

$$\mathbf{R} = \mathbf{X}\mathbf{X}^H = \mathbf{A}\mathbf{A}^H + \sigma^2 I$$

- Using the SVD of $\mathbf{A}$ we find

$$\mathbf{X}\mathbf{X}^H = \mathbf{U}\left[\mathbf{\Sigma}\cancel{\mathbf{V}^H\mathbf{V}}\mathbf{\Sigma}\right]\mathbf{U}^H + \sigma^2 I \; = \mathbf{U}\left[\mathbf{\Sigma}^2 + \sigma^2 I\right]\mathbf{U}^H$$

which is the covariance matrix EVD

# Blind Source Separation

- For simplicity assume $\mathbf{\Sigma}^2 + \sigma^2 I \approx \mathbf{\Sigma}^2$ i.e. high SNR then

$$\mathbf{X}\mathbf{X}^H = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^H$$

- So the covariance matrix gives us $\mathbf{U}$ and $\mathbf{\Sigma}$, and all we have to do is to find a way to calculate $\mathbf{V}$.
- Now note that

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathcal{N} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H\mathbf{S} + \mathcal{N}$$

- Thus, assuming $\mathbf{\Sigma}^{-1}$ exists, define

$$\mathbf{Y} \equiv \mathbf{\Sigma}^{-1}\mathbf{U}^H\mathbf{X} = \mathbf{V}^H\mathbf{S} + \tilde{\mathcal{N}}$$

where $\tilde{\mathcal{N}} = \mathbf{\Sigma}^{-1}\mathbf{U}^H\mathcal{N}$ is a noise term

# Blind Source Separation

- We have

$$\mathbf{Y} = \mathbf{V}^H \mathbf{S} + \tilde{\mathcal{N}}$$

- so $\mathbf{S}$ could be extracted from $\mathbf{Y}$ if we knew $\mathbf{V}^H$
- Then

$$\hat{\mathbf{S}} = \left(\mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^H\right)\mathbf{X}$$

- cf. bank of beamformers

$$\hat{\mathbf{S}} = \begin{bmatrix} \mathbf{w_1}^H \\ \vdots \\ \mathbf{w_N}^H \end{bmatrix} \mathbf{X}$$

- Blind signal separation is limited by what a bank of beamformers can do e.g. $N$ sensors $\rightarrow N-1$ nulls
- Recall I said "Performance limited by 'optimum' filter" on slide 3

# Blind Source Separation

- How to estimate $\mathbf{V}^H$?
  NB
  $$\mathbf{Y}\mathbf{Y}^H = \mathbf{V}^H\mathbf{S}\mathbf{S}^H\mathbf{V} + \sigma^2\mathbf{\Sigma}^{-2}$$
  but $\mathbf{S}\mathbf{S}^H = I$ so
  $$\mathbf{Y}\mathbf{Y}^H = I + \sigma^2\mathbf{\Sigma}^{-2}$$
  i.e. not dependent on $\mathbf{V}^H$ so the second order statistics of $\mathbf{Y}$ will not help us estimate $\mathbf{V}^H$
- Can however use higher order statistics
- Can also use nonlinear cost function
- At this point we have stop manipulating matrices and deal with actual time series data

# Blind Source Separation

- E.g. 'FastICA' - iteration to minimise 'negentropy'

$$J(Y) = H(\mathbf{Y}_{\mathbf{Gauss}}) - H(\mathbf{Y})$$

- $\mathbf{Y}_{\mathbf{Gauss}}$ is Gaussian data with same covariance matrix as $\mathbf{Y}$, $H(\mathbf{Y})$ is the entropy of $\mathbf{Y}$

$$H(Y) = -\int p_Y(y) \log(p_Y(y))dy$$

- Iteration

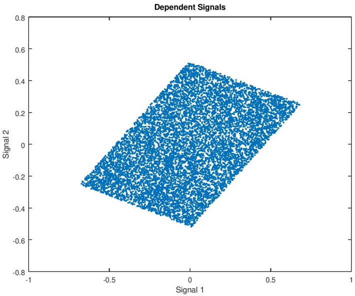
$$\mathbf{V}_{k+1} = G\left(\mathbf{V}_k^H \mathbf{Y}\right)^H \mathbf{Y} - G'\left(\mathbf{V}_k^H \mathbf{Y}\right)^H \mathbf{V}_k$$

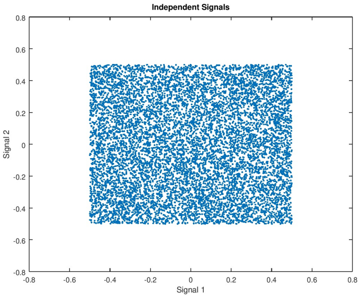
$$G(v) = \tanh(\alpha v), v \exp(-v^2/2), \text{or } v^3$$

where $1 \leq \alpha \leq 2$

# Blind Source Separation

- 'BLISS': a more overt exploitation of higher order statistics
- Statistical independence $P(x, y) = P(x)P(y)$
- Scatter diagram




- Calculate rotation (i.e. unitary matrix $\mathbf{V}$) to align scatter plot with axes

# Blind Source Separation

- Estimating the 'hidden' rotation matrix

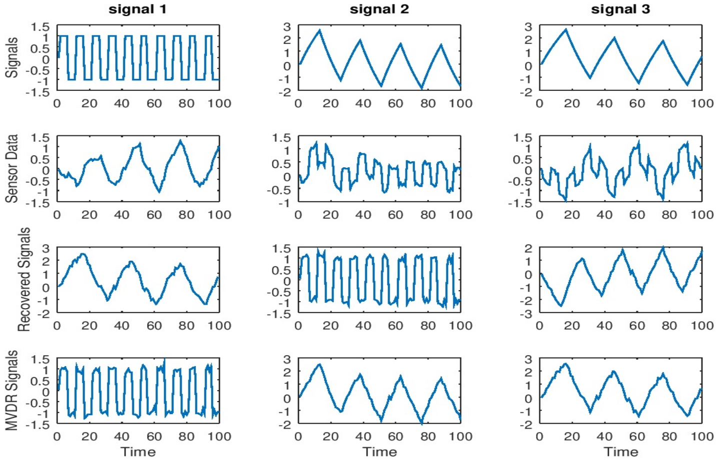
$$\mathbf{Y} = \mathbf{V}^H\mathbf{S} + \tilde{\mathcal{N}}$$

- Loop through all pairs of signals
- Rotate (with $\mathbf{Q_i}$ say) to align with axes
- Repeat until rotation angle is below a threshold

$$\mathbf{Q_n}\mathbf{Q_{n-1}}...\mathbf{Q_1}\mathbf{Y} = \hat{\mathbf{S}}$$

- Can show that $\hat{\mathbf{S}}$ is $\mathbf{S}$ up to scaling and permutation of the signal order

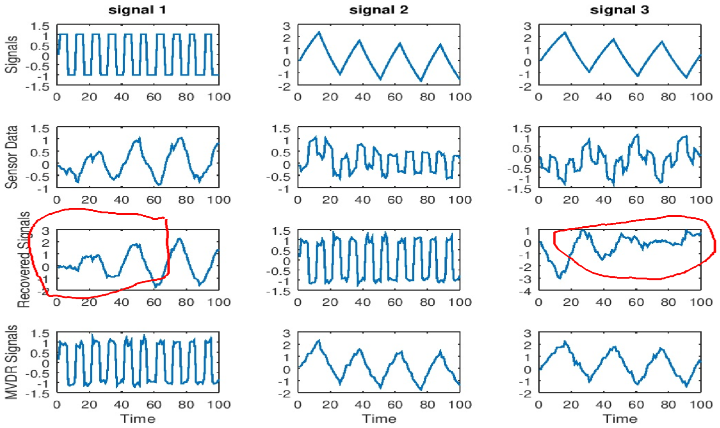
# Blind Source Separation




- 3 signals,3 sensors, SNR = 20dB
- MVDR as benchmark (recall: has access to more information)

# Blind Source Separation

- Need more data to calculate higher-order statistics




- Previous plot: 1000 data samples, This plot: 100 data samples
- NB. MVDR largely unaffected

# Summary

- Signal Separation: need a ‘filter’ and to estimate parameters
  Performance limited by ‘optimum’ filter
- Non-adaptive beamforming
  Good optimisation algorithms
- Adaptive signal processing for beamforming
  Constrain direction of main beam, reduce everything else
  Weight jitter, calibration errors
  Lots of linear algebra
- Blind source separation
  Higher-order statistics or nonlinear optimisation
  Needed more data to get good result

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