Estimation of Source-Sensor Responses from Sensor Second Order Statistics

Stephan Weiss, Fraser K. Coutts, Ian K. Proudler, Adam Getty

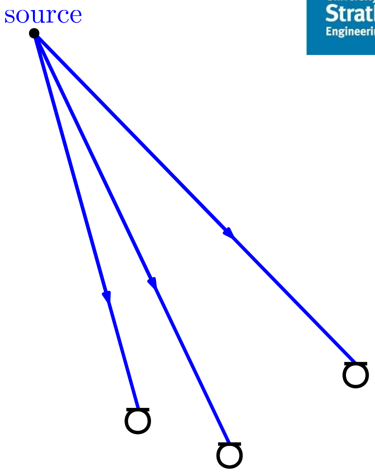
Dept. of EEE, Univ. of Strathclyde, Glasgow, Scotland

Thanks to: Nick Goddard, Dstl

UDRC Underwater Theme Day, 25 March 2021

This work was supported the Engineering and Physical Sciences Research Council (EPSRC) Grant number EP/S000631/1 and the MOD University Defence Research Collaboration in Signal Processing.

Presentation Overview



Presentation Overview

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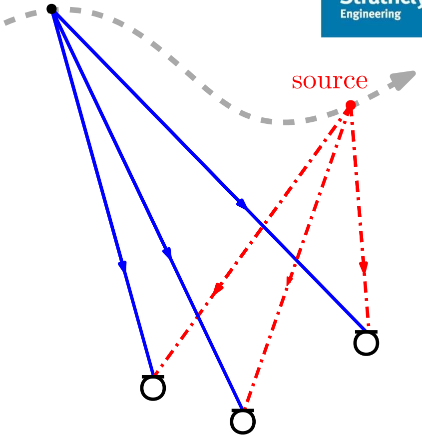
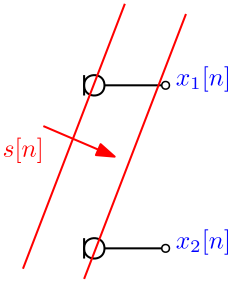
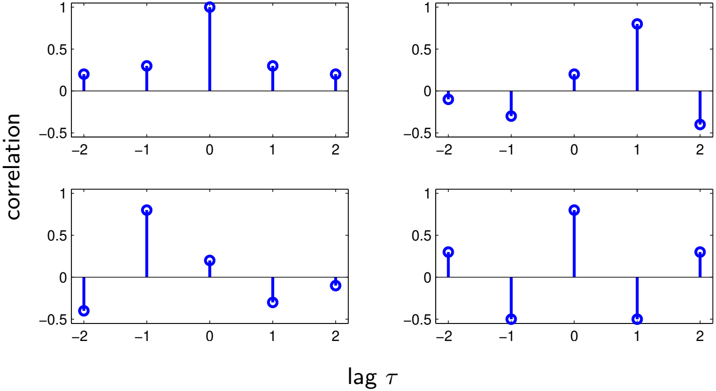


Space-Time Covariance

- We have M sensor signals organised in $\mathbf{x}[n] \in \mathbb{C}^M$;
- to take the broadband nature of signals into account, we consider a lag τ;
- space-time covariance matrix $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\rm H}[n-\tau]\}$;





Cross-Spectral Density Matrix

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- CSD matrix forms a z-transform pair with the space-time covariance matrix,

$$\boldsymbol{R}(z) = \sum_{\tau} \mathbf{R}[\tau] z^{-\tau} \qquad \text{or} \qquad \boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau] \; ;$$

- symmetry of $\mathbf{R}[\tau] \;\longrightarrow\; \boldsymbol{R}(z)$ is parahermitian:

$$\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{P}}(z) = \boldsymbol{R}^{\mathrm{H}}(1/z^{*}) \; ;$$

  (Hermitian transposition and time reversal)

- link to a narrowband covariance at normalised angular freq. Ω_k,

$$\mathbf{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \boldsymbol{R}(z)\big|_{z=\mathrm{e}^{\mathrm{j}\Omega_k}}$$

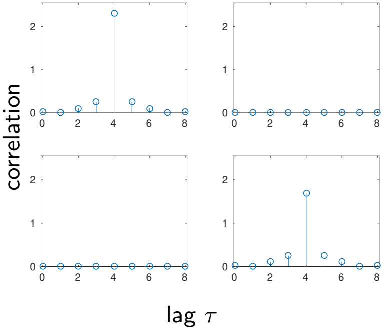
- many optimal (narowband!) methods are based on decompositions such as the EVD: $\mathbf{R}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathrm{H}}$.

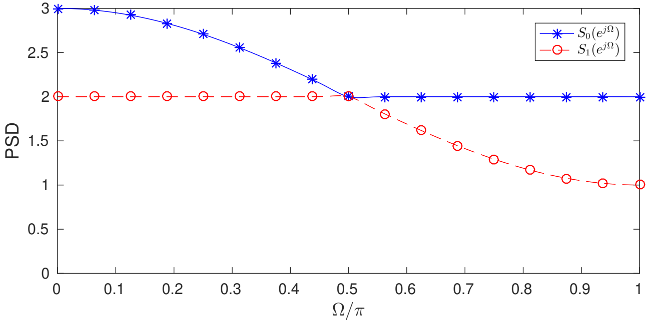
McWhirter Decomposition

- John McWhirter et al. (2007): polynomial eigenvalue decomposition of a parahermitian matrix:

$$\boldsymbol{R}(z) \approx \boldsymbol{Q}(z)\boldsymbol{\Lambda}(z)\boldsymbol{Q}^{\mathrm{P}}(z)$$

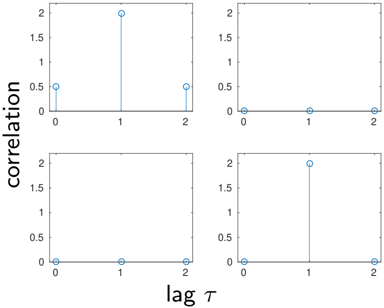
- paraunitary (i.e. lossless) matrix $\boldsymbol{Q}(z)$, s.t. $\boldsymbol{Q}(z)\boldsymbol{Q}^{\mathrm{P}}(z) = \mathbf{I}$;
- diagonal and spectrally majorised $\boldsymbol{\Lambda}(z)$:

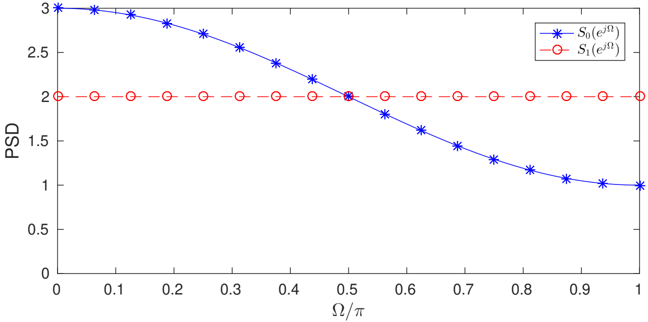




Parahermitian Matrix EVD (PEVD)

- For $\boldsymbol{R}(\mathrm{e}^{\mathrm{j}\Omega})$ analytic, in almost all cases there exist analytic eigenvectors $\boldsymbol{\Gamma}(\mathrm{e}^{\mathrm{j}\Omega})$ and analytic eigenvalues $\boldsymbol{U}(\mathrm{e}^{\mathrm{j}\Omega})$, s.t. $\boldsymbol{R}(z) = \boldsymbol{U}(z)\boldsymbol{\Gamma}(z)\boldsymbol{U}^{\mathrm{P}}(z)$;





- eigenvalues are unique, eigenvectors can be modified by arbitrary allpass filters $H(z)$ (s.t. $H(z)H^{\mathrm{P}}(z) = 1$),

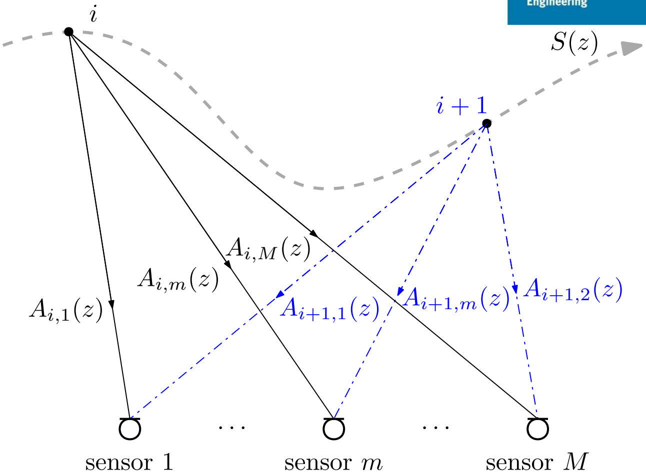
$$\boldsymbol{R}(z)\boldsymbol{u}(z)H(z) = \gamma(z)\boldsymbol{u}(z)H(z) \; .$$

Source-Sensor Transfer Functions

- We take M-array measurements of a single source:
- 2nd order stats:

$$
\begin{aligned}
\boldsymbol{R}_i(z) &= S(z)\boldsymbol{a}_i(z)\boldsymbol{a}_i^{\mathrm{P}}(z) \\
&= \gamma_i(z)\boldsymbol{u}_i(z)\boldsymbol{u}_i^{\mathrm{P}}
\end{aligned}
$$

- we can also divide a single array measurement into at least two subarrays with associated CSD matrices.



Transfer Functions and PEVD

- 2nd order stats: $\boldsymbol{R}_i(z) = S(z)\boldsymbol{a}_i(z)\boldsymbol{a}_i^{\mathrm{P}}(z) = \gamma_{i,m}(z)\boldsymbol{u}_i(z)\boldsymbol{u}_i^{\mathrm{P}}$;
- difference: $\boldsymbol{u}_i(z)$ is normal, $\boldsymbol{u}_i^{\mathrm{P}}(z)\boldsymbol{u}_i(z) = 1$, while $\boldsymbol{a}_i(z)$ is not:

$$\boldsymbol{a}_i^{\mathrm{P}}(z)\boldsymbol{a}_i(z) = A_{i,(-)}(z)A_{i,(+)}(z) = A_{i,(+)}^{\mathrm{P}}(z)A_{i,(+)}(z)$$

  with minimum-phase $A_{(+)}(z)$;
- therefore:

$$H_i(z)\boldsymbol{u}_i(z) = \frac{\boldsymbol{a}_i(z)}{A_{i,(+)}(z)}$$

$$\gamma_i(z) = A_{i,(+)}(z)S(z)A_{i,(+)}^{\mathrm{P}}(z) \ ,$$

- from a single measurement $\boldsymbol{R}_i(z)$, we cannot say anything about $\boldsymbol{a}_i(z)$ or $S(z)$.

Multiple Measurements

- If we have several measurements $i = 1 \ldots I$:

$$H_i(z)\boldsymbol{u}_i(z) = \frac{\boldsymbol{a}_i(z)}{A_{i,(+)}(z)}$$

$$\gamma_i(z) = A_{i,(+)}(z)S(z)A^{\mathrm{P}}_{i,(+)}(z) \ ,$$

- we can extract $S(z)$ as the greatest common divisor

$$\hat{S}(z) = \mathsf{GCD}\{\gamma_1(z) \ \ldots \ \gamma_I(z)\} \ ;$$

- we can then extract the terms $A_{i,(+)}(z)$, and hence determine the vectors $\boldsymbol{a}_i(z)$ save of an arbitrary phase response due to the allpass $H_i(z)$:

$$\boldsymbol{a}_i(z) = A_{i,(+)}(z)H_i(z)\boldsymbol{u}_i(z) \ .$$

Alternative DFT Domain Attempt

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- As an alternative, we take measurements in independent frequency bins $\Omega_k = \frac{2\pi k}{K}$:

$$\begin{aligned}\mathbf{R}_{i,k} = \boldsymbol{R}_i(\mathrm{e}^{\mathrm{j}\Omega_k}) &= \boldsymbol{a}_i(\mathrm{e}^{\mathrm{j}\Omega_k})S(\mathrm{e}^{\mathrm{j}\Omega_k})\boldsymbol{a}_i^{\mathrm{H}}(\mathrm{e}^{\mathrm{j}\Omega_k}) \\ &= \mathbf{q}_{i,k}\lambda_{i,k}\mathbf{q}_{i,k}^{\mathrm{H}} \ .\end{aligned}$$

- principal eigenvectors and eigenvalues for the measurement campaigns are

$$\begin{aligned}\mathbf{q}_{i,k} &= \frac{\boldsymbol{a}_i(\mathrm{e}^{\mathrm{j}\Omega_k})}{|\boldsymbol{a}_i(\mathrm{e}^{\mathrm{j}\Omega_k})|} \ , \\ \lambda_{i,k} &= S(\mathrm{e}^{\mathrm{j}\Omega_k})|\boldsymbol{a}_i(\mathrm{e}^{\mathrm{j}\Omega_k})|^2 \ .\end{aligned}$$

- because of the normalisation, nothing can be extracted about the source or the transfer functions.

Numerical Example

- Source with power spectral density

$$S(z) = \frac{1}{2}z + \frac{5}{4} + \frac{1}{2}z^{-1}$$

- vector of transfer functions during campaign $i = 1$:

$$\boldsymbol{a}_1(z) = \begin{bmatrix} 1 & + & \frac{1}{2}z^{-1} \\ \frac{3}{4} & - & \frac{1}{2}z^{-1} \end{bmatrix}$$

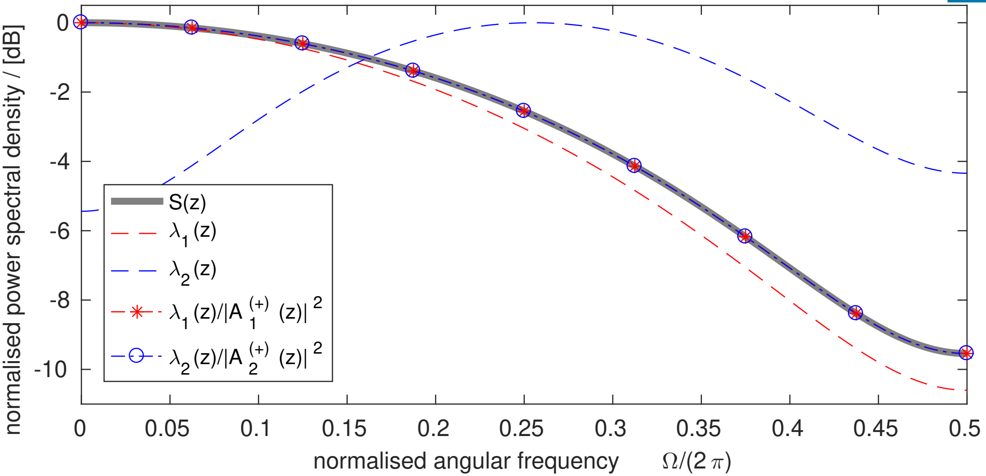
- vector of transfer functions during campaign $i = 2$:

$$\boldsymbol{a}_2(z) = \begin{bmatrix} \frac{4}{5} & - & \frac{1}{2}z^{-1} \\ -\frac{1}{2} & + & z^{-1} \end{bmatrix} ;$$

- based on these: PEVD computations for $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$, and GCD calculation based on eigenvalues.

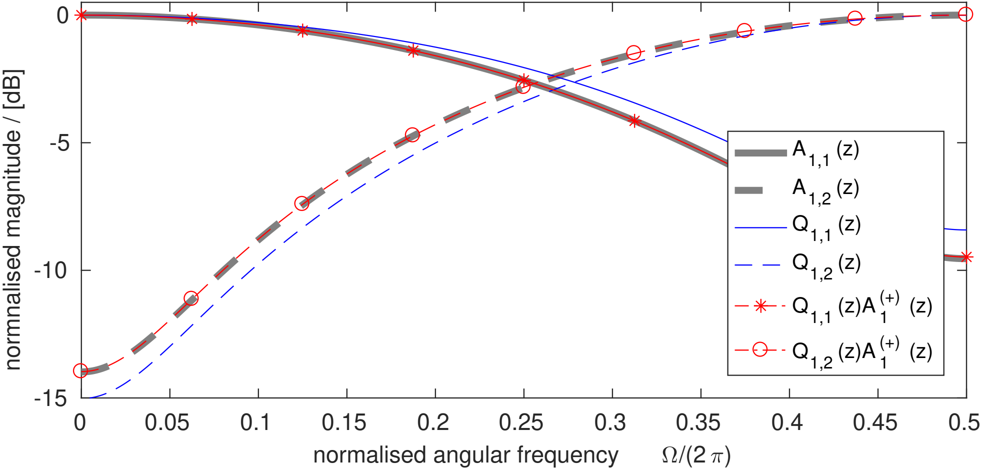
Numerical Results — Source PSD

- Eigenvalues / source PSD for both measurements $i = \{1, 2\}$:



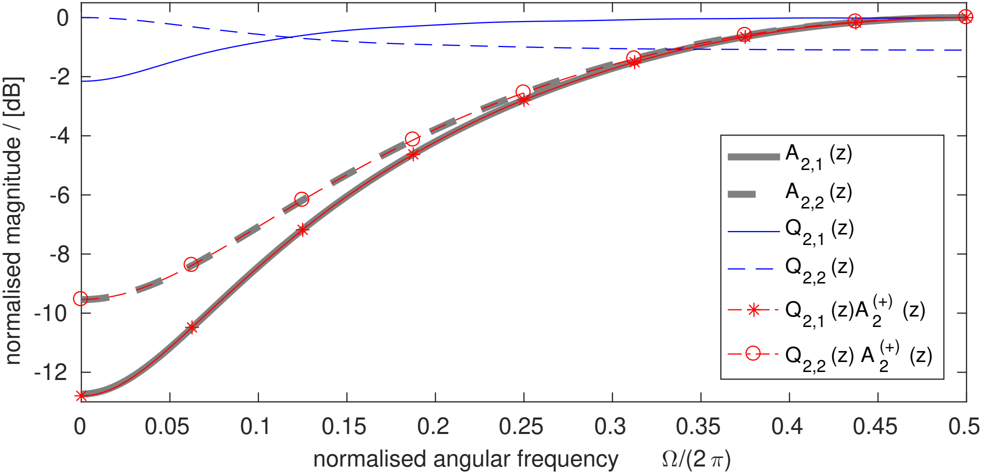
Numerical Result — Magnitude Responses I

- Eigenvectors / magnitude response for measurement $i = \{1\}$:



Numerical Result — Magnitude Responses II

- Eigenvectors / magnitude response for measurement $i = \{2\}$:



Summary and Reflection

- We can extract the source PSD and the magnitude responses once we have at least two measurements;
- these measurements can be taken in time or space;
- an independent frequency bin approach does not yield anything;
- the polynomial approach rests on an accurate parahermitian EVD, and an accurate root finding / GCD algorithm;
- root finding is numerically challenging: research since Euclid (300BC), with robust root-finding methods still on-going (monic polynomial subtraction, Gröbner bases, algebraic geometry);
- the problem can be somewhat alleviated by lower-order factorisations by a new set of analytic PEVD algorithms (SSPD'20 and IEEE TSP'21);
- nevertheless the approach gives a glimpse of the type of advantages that a coherent broadband approach can offer.