Super-Resolved (Gridless) Wideband DoA Estimation

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Outline

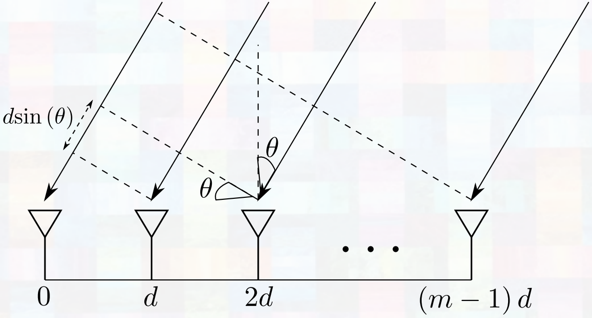
DoA: Direction of arrival

1. Narrow-band DoA estimation
2. Wide-band DoA estimation
3. Our approach:

   Antenna-Frequency Interwoven Steering and Estimate (Afise)

Applications:
Radar, sonar, acoustic sensing, spectral estimation, seismology, astronomy, medical imaging, etc.

System Model



- Uniform linear array + far field

Steering Vectors and Subspaces

- Steering vectors

  (d has been normalised by the speed of light c)

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{-j\omega d \sin\theta} \\ \vdots \\ e^{-j(M-1)\omega d \sin\theta} \end{bmatrix}$$

- Steering matrix

$$\begin{aligned} \boldsymbol{y} &= \sum_{l=1}^{L} \boldsymbol{a}(\theta_l) s_l + \boldsymbol{v} \\ &= \underbrace{\left[\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_L)\right]}_{\boldsymbol{A}(\boldsymbol{\theta})} \boldsymbol{s} + \boldsymbol{v} \end{aligned}$$

MUSIC [Schmidt 1986]

- Empirical covariance matrix

$$\hat{\mathbf{\Sigma}}_Y \overset{N\to\infty}{\to} \boldsymbol{A}\Sigma_S\boldsymbol{A}^H + \sigma^2\boldsymbol{I}$$

- Eigen-decomposition for signal/noise subspace $\boldsymbol{U}_Y/\boldsymbol{U}_Y^{\perp}$
- Extract DoA from noise subspace

$$\arg\min_{\phi}\ \boldsymbol{a}(\phi)^H\boldsymbol{U}_Y^{\perp}\boldsymbol{U}_Y^{\perp,T}\boldsymbol{a}(\phi)$$

✎ Signal subspace span$(\boldsymbol{U}_Y)$ may not be a valid steering subspace

ESPRIT [Roy & Kailath 1989]

- Invariance property:

$$\boldsymbol{a}(\alpha) := \begin{bmatrix} 1 \\ e^{-j\alpha} \\ \vdots \\ e^{-j(M-1)\alpha} \end{bmatrix} \Rightarrow \text{span}(\boldsymbol{a}(\alpha)_{1:M-1}) = \text{span}(\boldsymbol{a}(\alpha)_{2:M})$$

✓ Generalised eigen-decomposition:

Signal subspace $\text{span}(\boldsymbol{U}_Y) =$ a valid steering subspace

Atomic Norm Minimisation [Candès & Fernandez-Granda 2014; Tang et al., 2013]

- Low-rank Toeplitz matrix

$$\boldsymbol{A}(\alpha)\boldsymbol{A}(\alpha)^H = \underbrace{\begin{bmatrix} u_1 & u_2^* & \cdots & u_M^* \\ u_2 & u_1 & \cdots & u_{M-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ u_M & u_{M-1} & \cdots & u_1 \end{bmatrix}}_{\text{Toeplitz matrix}}$$

✓ Convex Optimisation

$$\min_{\boldsymbol{u},t,\boldsymbol{x}} \|\boldsymbol{W}\|_* + \frac{1}{\sigma^2}\|\boldsymbol{y} - \boldsymbol{x}\|_2^2$$

$$\text{s.t. } \boldsymbol{W} = \begin{bmatrix} \text{Toep}(\boldsymbol{u}) & \boldsymbol{x} \\ \boldsymbol{x}^H & t \end{bmatrix} \succeq 0$$

✎ Minimum separation requirement

Orthogonal Atomic Norm Minimisation [Chen & Chi 2014; Xu et al., 2018]

- Low rank Hankel matrix

$$\boldsymbol{A}(\alpha)\boldsymbol{A}(\alpha)^T = \underbrace{\begin{bmatrix} h_1 & h_2 & \cdots & u_M \\ h_2 & u_3 & \cdots & u_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_M & u_{M+1} & \cdots & u_{2M-1} \end{bmatrix}}_{\text{Hankel matrix}}$$

✓ Convex optimisation

$$\min_{\boldsymbol{x}} \|\boldsymbol{W}\|_* + \frac{1}{\sigma^2}\|\boldsymbol{y} - \boldsymbol{x}\|_2^2$$
$$\text{s.t. } \boldsymbol{W} = \mathcal{H}(\boldsymbol{x})$$

✓ Remove minimum separation requirement

- Performance depends on the design of the Hankel matrix

Wideband DoA Estimation and Challenges

- Steering vector as a nonlinear function of the frequency ω

$$\boldsymbol{a}(\theta;\omega) = \begin{bmatrix} \vdots \\ e^{-j(m-1)\omega d \sin\theta} \\ \vdots \end{bmatrix}$$

- Steering subspace as a nonlinear function of the frequency ω

$$\mathrm{span}(\boldsymbol{A}(\theta;\omega_0)) \neq \mathrm{span}(\boldsymbol{A}(\theta;\omega_1))$$

- Key challenge for wideband DoA estimation
  - Joint estimation using information at different frequencies

Incoherent Signal Subspace Method (ISSM)

[Wax et al., 1984; Hung & Kaveh 1990]

- 'Average' of narrowband estimates

$$\hat{\theta} = \arg\min_{\theta} \sum_{i=0}^{N-1} \boldsymbol{a}^H(\theta;\omega_n)\boldsymbol{U}_n^{\perp}\boldsymbol{U}_n^{\perp,H}\boldsymbol{a}(\theta;\omega_n)$$

✎ Two stages: narrowband subspace estimate followed by 'average'.

TOPS [Yoon et al., 2006]

- Exact transform between steering vectors at different frequencies

$$\begin{aligned}\omega_n d \sin\theta &= \omega_0 d \sin\theta + (\omega_n - \omega_0) d \sin\theta \\ &= \omega_0 d \sin\theta + \Delta_\omega d \sin\theta\end{aligned}$$

and hence

$$\boldsymbol{a}(\theta;\omega_n) = \mathrm{diag}\left(\boldsymbol{a}(\theta;\Delta_\omega)\right)\boldsymbol{a}(\theta;\omega_0)$$

- Null space test

$$\left(\mathrm{diag}\left(\boldsymbol{a}(\phi;\Delta_\omega)\right)\boldsymbol{U}_0\right)^T \boldsymbol{U}_n^\perp \text{ is rank deficient, } \forall n$$

when $\phi \in \{\theta_1, \cdots, \theta_L\}$

✎ Two stages: narrowband subspace estimate followed by a 'test'.

Coherent Signal Subspace Method (CSSM)

[Wang & Kaveh 1985; Hung & Kaveh 1988]

- Set a focusing (reference) frequency ω_0.
- Transform steering matrices using a linear approximation

$$\boldsymbol{A}(\theta;\omega_0) \approx \boldsymbol{T}_n \boldsymbol{A}(\theta;\omega_n)$$

Define a general covariance matrix

$$\begin{aligned}\boldsymbol{\Sigma}_y^g &= \sum_n \boldsymbol{T}_n \boldsymbol{\Sigma}_y(\omega_n) \boldsymbol{T}_n^H \\ &= \sum_n \boldsymbol{A}(\theta;\omega_0) \boldsymbol{\Sigma}_s(\omega_n) \boldsymbol{A}(\theta;\omega_0)^H + \sum_n \boldsymbol{T}_n \boldsymbol{\Sigma}_v(\omega_n) \boldsymbol{T}_n^H\end{aligned}$$

✓ Steering subspace joint estimation

✎ The approximation is θ dependent

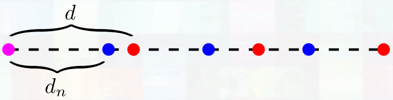
Virtual Array and Spatial Interpolation

[Krolik & Swingler 1990; Raimondi et al., 2016]

- Create a virtual array at frequency ω_n:

$$\omega_n d_n = \omega_0 d \quad \Rightarrow \quad d_n = \frac{\omega_0}{\omega_n} d$$

Virtual array: Resampling $\boldsymbol{y}(\omega_n)$ with spatial 'rate' d_n



✓ Steering subspace joint estimation

✎ Noise gets interpolated as well

Virtual Array via Jocobi-Anger Expansion

[Wang et al., 2019, Wang et al., 2020]

- Virtual array via Jocobi-Anger expansion

$$e^{j\omega d \sin\theta} = \sum_{k=-\infty}^{\infty} J_k(\omega d) e^{jk\theta}$$

where J_k is the k-th Bessel function of the first kind

✓ Steering subspace joint estimation

✓ It works for non-uniform arrays

✎ The infinite series needs to be truncated as an approximation

✎ Non-unit gain antennas

Parahermitian Matrix EVD and Polynomial MUSIC

[Alrmah et al., 2011; Alrmah et al., 2012; Weiss et al., 2013; Alrmah et al., 2014]

- Space-time covariance matrix: $\mathsf{R}(\tau) := \mathsf{E}[\boldsymbol{y}(n)\boldsymbol{y}^H(n-\tau)]$
- Cross spectral density: $\boldsymbol{R}(z) := \sum_\tau \mathsf{R}(\tau) z^{-\tau}$
- Finite length Laurent polynomial approximation
  $\Rightarrow$ Polynomial EVD or McWhirter decomposition
  $$\boldsymbol{R}(z) = \hat{\boldsymbol{U}}(z)\hat{\boldsymbol{\Lambda}}(z)\hat{\boldsymbol{U}}^P(z)$$
- Polynomial MUSIC
  $$\boldsymbol{s}^P(z)\boldsymbol{U}^\perp(z)\boldsymbol{U}^{\perp,P}(z)\boldsymbol{s}(z)$$
  where $\boldsymbol{s}(z)$ is the z-transform of $\boldsymbol{s}(n) = \frac{1}{\sqrt{M}}[\cdots, s(n-\tau_m), \cdots]^T$.

✎ Approximation of $\mathsf{R}(\tau)$ and $\boldsymbol{R}(z)$ in practice

Our Approach

- Design considerations for complicated EM environment
  - ♥ Fleeting signals (agility):
    cannot rely on covariance matrix estimation
  - ♥ Low SNR (resilience): need to avoid approximation
  - ♥ Fast algorithm for real-time applications (agility)

Our Approach

- Design considerations for complicated EM environment
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    cannot rely on covariance matrix estimation
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- A uniform grid in frequency
  - $[\omega_L, \omega_H] \Rightarrow \{\omega_L, \omega_L + \Delta_\omega, \cdots, \omega_H\}$
  - Already default in DFT

Antenna-Frequency Interwoven Steering

$$\boldsymbol{Y} = [\boldsymbol{y}(\omega_1), \cdots, \boldsymbol{y}(\omega_N)] = \boldsymbol{X} + \boldsymbol{V}$$

where

$$X_{m,n} = \sum_l s_n^l e^{-j(m-1)(\omega_1+(n-1)\Delta_\omega)\sin\theta_l}$$

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- Antenna steering

$$\boldsymbol{X}_{:,n} = \sum_l s_n^l \left[\cdots, e^{-j(m-1)\omega_n \sin\theta_l}, \cdots\right]^T$$

Antenna-Frequency Interwoven Steering

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- Antenna steering
$$\boldsymbol{X}_{:,n} = \sum_l s_n^l \left[\cdots, e^{-j(m-1)\omega_n \sin\theta_l}, \cdots\right]^T$$
- Frequency steering
$$\boldsymbol{X}_{m,:} = \sum_l \boldsymbol{s}^{l,T} \odot \left[\cdots, e^{-j((m-1)\omega_1+(n-1)(m-1)\Delta_\omega)\sin\theta_l}, \cdots\right]$$
Information at different frequencies are now linked via frequency steering vectors.

Antenna-Frequency Interwoven Steering and Estimate (Afise)

- Trilinear inverse problem

Observed data

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{V}$$

$$= \sum_{l=1}^{L} \operatorname{diag}(\boldsymbol{a}(\alpha_l)) \, \boldsymbol{V}(\boldsymbol{a}(\beta_l)) \, \operatorname{diag}(\boldsymbol{s}^l) + \boldsymbol{V}$$

Unknown number of sources

Unknown antenna steering

Unknown frequency steering

Unknown frequency coefficients

Afise: Preliminary Results

- ✓ Designed a convex Optimisation
- ✓ Achieved steering subspace joint estimation
- Much more work needs to be done!

Thank You and Questions!