Radar Waveform Libraries Using Fractional Fourier Transform

Carmine Clemente *, Ivan Shorokhov*, Ian Proudler†, John J. Soraghan*
*Centre for Excellence in Signal and Image Processing, University of Strathclyde, Glasgow, United Kingdom
Email:carmine.clemente@strath.ac.uk, wsb12203@uni.strath.ac.uk, j.soraghan@strath.ac.uk
† School of Electronic, Electrical and Systems Engineering, Loughborough University, Leicestershire, UK
Email: i.k.proudler@lboro.ac.uk

***Abstract*—Modern radar systems, such as co-located or distributed MIMO radar, often operate in very hostile environments. In such a scenario the selection of the best waveform to be used is an important aspect in order to maximize the performance. In this paper we introduce the use of the Fractional Fourier Transform as a tool to generate libraries of phase coded waveforms. The analysis of the performance of the generated libraries demonstrates how the novel approach can introduce benefits for specific applications with a relatively simple effort and with limited or no extra resource requirement.**

I. Introduction

In the modern battlefield scenario radar systems typically operate in a congested electromagnetic environment and often with severe constrains in terms of interference mitigation, frequency occupancy, security and performance. Coexistence of different systems for different applications, high accuracy in target detection, tracking and recognition, low probability of intercept, jamming and MIMO radar are example where a hostile environment, from an electromagnetic point of view, can cause dramatic consequences on the overall performance. It is in this scenario that the selection of the most suitable waveform can play an important role. Fixed and adaptive radar waveform design has been widely investigated [1], [2], providing waveforms that can suit different applications, however each of them presents a trade-off between some of their characteristics such as range resolution versus side lobe levels. The opportunity to form novel libraries of waveforms that are able to maintain the same or higher level of performance is of interest to the radar community and other related disciplines.[3].

In this paper the use of the fractional Fourier Transform (FrFT) based phase coded waveforms are introduced to generate new family of waveform libraries. The FrFT, is a generalization of the Fourier transform and has already been applied in radar signal processing [4] and OFDM modulation [5] demonstrating the potential of this signal processing tool for various applications [6]. In particular in [5] the FrFT has been used to modulate OFDM signals providing improvement in the overall bit error rate (BER), this improvement was mainly due to a reduction of the overall interference between the novel fractional sub-carriers, increasing the signal to interference ratio leading to an improvement in BER.

In our approach for the generation of the novel radar waveform libraries the FrFT is applied to the waveforms (e.g. the code sequence). We analyse the ambiguity functions [7] of the generated waveforms in order to quantify the performance provided by the proposed libraries. A deeper analysis about the orthogonality and reuse of the proposed library is presented in [8].

The remainder of the paper is organized as follows. Section II introduces the fractional Fourier transform, while in Section III the novel modulation approach and the relationship between the ambiguity functions are introduced. The analysis the performance for sample novel libraries is presented in Section IV, while Section V concludes the paper.

II. Fractional Fourier Transform

A Fourier transformation (FT) maps a one-dimensional time signal $x(t)$ into a one-dimensional frequency function $X(f)$, the signal spectrum. The Fourier transform operator can be visualized as a change in representation of the signal corresponding to a counter clockwise rotation of the axis by an angle $\pi/2$ in the time-frequency plane . Although the Fourier transform provides the spectral content of the signal, it fails to indicate the time location of the spectral components, which is of great importance when non-stationary or time-variant signals are considered. In order to describe and analyse such signals, time-frequency representations (TFRs) are used. A TFR maps a one-dimensional time signal into a two-dimensional function of time and frequency. The fractional Fourier transform which belongs to the class of linear TFRs was introduced by Namias in 1980 [9], then rediscovered in optics [10], [11], [6], [12] and introduced to the signal processing community by Almeida in 1994 [13].

The fractional Fourier transform, which is a generalization of the ordinary Fourier transform, can be considered as a rotation by an arbitrary angle in the time-frequency plane or a decomposition of the signal in terms of chirps. It also serves as an orthonormal signal representation for chirp signals and is also called rotational Fourier transform or angular Fourier transform [14]. The fractional Fourier transform is computed using the angle of rotation in the time-frequency plane as the fractional power of the ordinary Fourier transform. Letting $x(u)$ be an arbitrary signal, its ath-order FrFT is defined as

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[6]:

$$\mathbf{X}_a(u) = \int K_a(u,u')x(u')du' \tag{1}$$

where a is the fractional transformation order (corresponding to a rotation angle $\theta = a\frac{\pi}{2}$ with $a \in \mathbf{R}$) and $K_a(u,u')$ is the FrFT kernel and is defined as [6]:

$$K_a(u,u') = \begin{cases} A_0 \exp\left\{j\pi[(u^2+u'^2)\cot\theta - 2uu'\csc\theta]\right\} \\ \text{ if } \theta \text{ is not a multiple of } \pi \\ \delta(u-u') \text{ if } \theta \text{ is a multiple of } 2\pi \\ \delta(u+u') \text{ if } \theta+\pi \text{ is a multiple of } 2\pi \end{cases} \tag{2}$$

where $A_0 = \frac{e^{j\frac{\theta}{2}}}{\sqrt{j\sin\theta}}$

Equation (2) shows that for angles that are not multiples of π, the computation of the FrFT corresponds to the following steps:

1-A product by a chirp;

2-A Fourier transform (scaled by $\csc\theta$);

3-Another product by a chirp;

4-A product by a complex amplitude factor.

In summary, the FrFT is an invertible linear transform, continuous in the angle θ, which satisfies the basic conditions for it to be meaningful as a rotation in the time-frequency plane.

III. FrFT based Waveforms

In this section the new approach to obtain a novel library of waveforms is introduced.. To describe the performance of the proposed library the relationship between the ambiguity function of the canonical waveform and that of its generic FrFT is described.

The Fractional Fourier Transform introduced in Section II can be applied to common waveforms, such as phase modulated waveforms with different codes (i.e. Barker or P4 codes).

Let $s(u)$ be the canonical waveform (e.g. the traditional Barker 13 code) from which the Fractional Fourier transform library $\mathbf{S}_{a_i}(t)$ $i = 1, ..., L$ is obtained, by applying (1). Thus we define the fractional waveform library as:

$$\mathbf{S} = [S_{a_1}(t), S_{a_2}(t), \ldots, S_{a_L}(t)] \tag{3}$$

where $a_i \in [0,1]$, and L represents the total number of waveforms populating the library. Note that for $a_i = 0$ the canonical waveform is obtained. The value of L depends on different aspects such as the original waveform used, waveform reuse, orthogonality requirements and applications.

Understanding the properties of the resulting fractional modulated waveform is a critical aspect, and is accomplished using the ambiguity function (AF) of the signal $\mathbf{S}_a(t)$. The Ambiguity function for the signal $s(u)$, in the time - frequency plane (u,v), is defined as [7]

$$|\chi(u,v)|^2 = \left|\int_{-\infty}^{\infty} s\left(u+\frac{\varepsilon}{2}\right) s^*\left(u-\frac{\varepsilon}{2}\right) e^{-j\omega\varepsilon} d\varepsilon\right|^2 \tag{4}$$

By replacing in $s(\cdot)$ in (4) with $S_a(\cdot)$ we obtain the ambiguity function of the fractional waveform $|\chi_a(\cdot,\cdot)|^2$.

Almeida [13] and Djurovic [15] studied the relationship between the Fractional Fourier Transform and the Wigner Ville Distribution (WVD) $\chi(u,v)$. In particular they showed that the WVD of a signal and its Fractional Fourier Transform of order a are equivalent:

$$\chi_a(t,\omega) = \chi(u,v) \tag{5}$$

with the relationship between the variables described by the rotation $u = t\cos\theta + \omega\sin\theta$ and $v = -t\sin\theta + \omega\cos\theta$, and $\theta = a\pi/2$, thus

$$\int_{-\infty}^{\infty} S_a\left(t+\frac{\tau}{2}\right) S_a^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} s\left(u+\frac{\varepsilon}{2}\right) s^*\left(u-\frac{\varepsilon}{2}\right) e^{-jv\varepsilon} d\varepsilon \tag{6}$$

As the Ambiguity Function is the squared modulus of the Wigner Ville Distribution the relationship in (6) allows the expected ambiguity function for any fractional order a given the ambiguity function of the original signal $s(u)$ to be calculated.

IV. Analyis of the novel libraries

In this section the performance of two sample novel libraries are analysed. In particular the libraries for the conventional Frank and Barker codes are analysed. For each waveform library the analysis has been performed using the toolbox provided in [1] modified by introducing the Fractional Fourier Transform modulation for different fractional orders and extracting significant performance parameters.

In order to quantify the effectiveness of the novel libraries, for each waveform library, different performance parameters are analysed for the different fractional orders ($L = 100$):

- Delay resolution, computed as the -3 dB width of the 0 Doppler cut of the ambiguity function;
- Doppler resolution, computed as the -3 dB width of the 0 delay cut of the ambiguity function;
- Delay side lobe level, computed as the level of the first side lobe of the the 0 Doppler cut of the ambiguity function;
- Doppler side lobe level, computed as the level of the first side lobe of the the 0 delay cut of the ambiguity function;
- Modulated signal bandwidth, computed as the -3 dB width of the transmitted signal spectrum;
- Interfering power, computed as the power present outside the main lobe;
- Interfering power ratio, computed as the ratio between the power in the side lobes of the ambiguity function and the main lobe power.

These parameters are fundamental quality parameters of a radar waveform that aid in assessing the relative possible performance achievable and where the known trade-offs occur [1]. In our analysis we compare the above mentioned parameters with those from the original waveform. In particular the ratio between the parameters from the original waveform and those from the fractional waveform are computed. As all

the parameters requires to have small values to be considered as "good" an improvement can be measured for values < 1 ($< 10^0$ on the logarithmic scale used to present the results in this paper).

For each waveform library two configurations were considered, these as summarized in Table I. The parameter r is the number of samples per bit. Two values are considered. In the first case the value $r = 10$ as suggested in [1] is used, while in the second case an up-sampled waveform ($r = 300$) is used. Other parameters are $F * M * t_b$ representing the product between the maximum frequency shift, the sequence length and the bit duration, T representing the total sequence duration, while N and K are the total number of delay and Doppler shifts respectively. The analysis for different values of N and K is not presented as the AFs scale similarly for both the original and the fractional waveforms.

TABLE I
WAVEFORM ANALYSIS PARAMETERS

Parameters	Frank	Barker
r	10, 300	10, 300
$F * M * t_b$	24.96	20.02
T	1	1
N	160	130
K	50	50

A. Frank Code

The first analysis was performed using, as the canonical waveform, the canonical 16-element Frank code obtained with the phase shifts [1]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \pi/2 & \pi & 3\pi/2 & 0 & \dots \\ \dots & \pi 2 & 0 & \pi & 0 & 3\pi/2 & \pi & \pi/2 & \end{bmatrix} \quad (7)$$

In Figure 1 and Figure 2 the values of the quality parameter ratios with different fractional orders are shown. In Figure 1 the delay resolution and the interference factor are reduced over a wide range of fractional order values. However for other important parameters, such as Doppler resolution and Delay side lobe level, the fractional modulation results in lower performance compared to the canonical waveform ($\alpha = 0$). In contrast, as shown in 2, by increasing the number of samples per bit the fractional modulation results in improved delay resolution, whilst the interference level and the delay side lobe are both reduced for a significant range of fractional orders. Moreover for both the analyzed cases the occupied bandwidth is the same as that of the canonical waveform.

B. Barker Code

Barker Codes are probably the most famous family of phase codes. The sequence of Barker Code of length 13 has been used to perform our analysis. The Barker code considered is the sequence

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (8)$$

In Figure 3 and Figure 4 the values of the quality parameter ratios with different fractional orders are shown. Similar to

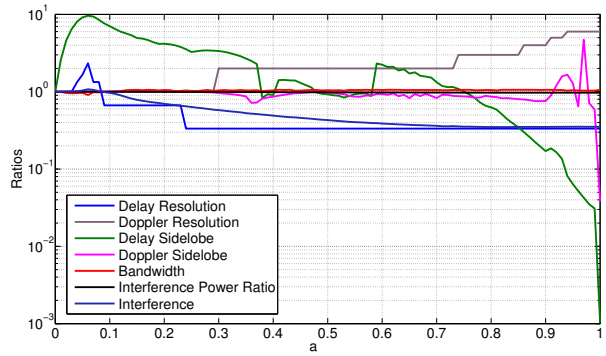

Fig. 1. Ratios of the ambiguity function quality parameter for the Frank Code library ($r = 10$).

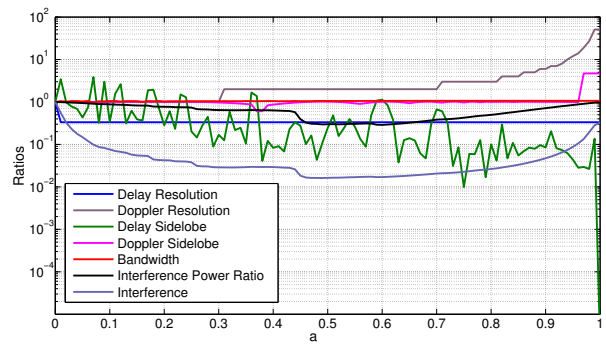

Fig. 2. Ratios of the ambiguity function quality parameter for the Frank Code library ($r = 300$).

the Frank code case, in Figure 3 shows that for a wide set of fractional order the delay resolution and the interference factor are reduced while for other parameters, the fractional modulation dose not result in improvements compared to the canonical waveform. Again, by increasing the number of samples per bit, the delay resolution improvement can be made to occupy a wider area of the fractional domain as well as achieving a reduction in the interference level and the level of delay side lobe - as shown in Figure 4. Moreover for both the analyzed cases the occupied bandwidth is the same as that of the canonical waveform.

The dependence of the performance from the number of samples per phase shift (the value of r) is an aspect that will require a deeper analysis and will be subject of future work. An important remark is that the limit of side lobe level and delay resolution of both the canonical Barker and Frank waveforms, dependent on the sequence length M, is no longer valid for the novel fractional libraries. In terms of computational complexity the novel libraries requires the computational overload of the FrFT which corresponds to 2 multiplications by a chirp and an FFT. However, the libraries can be pre computed and populate a Look Up Table.

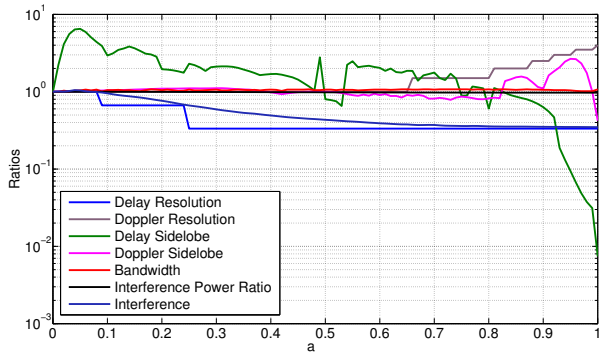

Fig. 3. Ratios of the ambiguity function quality parameter for the Barker Code library ($r = 10$).

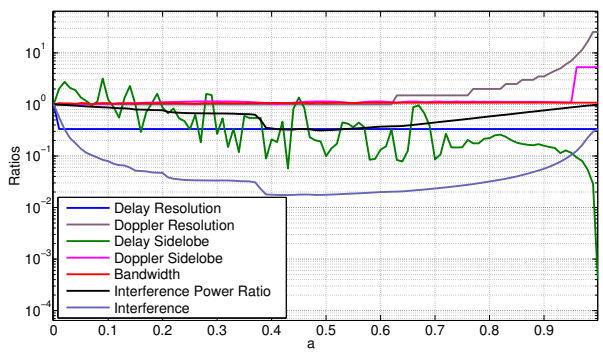

Fig. 4. Ratios of the ambiguity function quality parameter for the Barker Code library ($r = 300$).

V. Conclusions

This paper presented novel radar waveform libraries based on the use of the fractional Fourier transform modulation of well established radar waveforms. The novel libraries introduce significant advantages in terms of delay resolution, interference and side lobe level reduction. Potential applications of the novel libraries are in the field of frequency reuse, agile tracking-Doppler systems, low probability of intercept radar and MIMO radar systems.

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