Decision Theory

João Mota

UDRC-EURASIP Summer School 2022

Heriot-Watt University

Decision theory

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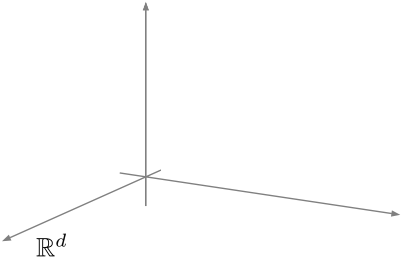
Problem

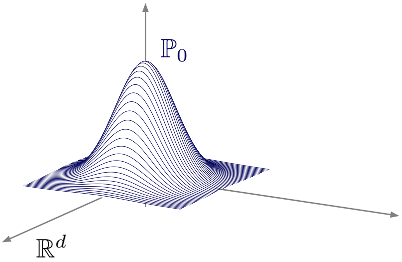
Decision theory



Problem

We observe $\boldsymbol{X} \in \mathbb{R}^d$

Decision theory

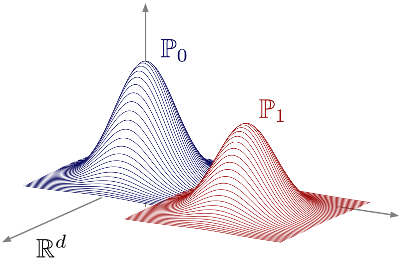


Problem

We observe $\boldsymbol{X} \in \mathbb{R}^d$

$\boldsymbol{X} \sim \mathbb{P}_0$

Decision theory



Problem

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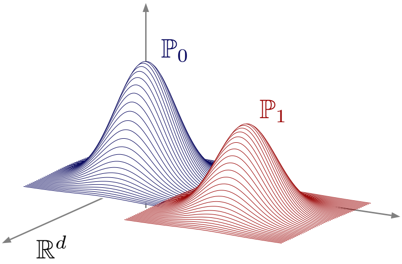
$\boldsymbol{X} \sim \mathbb{P}_0$ or $\boldsymbol{X} \sim \mathbb{P}_1$?

Decision theory



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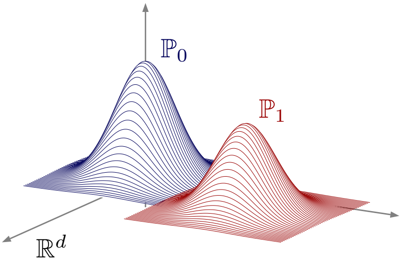

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In classical *decision theory*, we *know* the distributions $\mathbb{P}_0$ and $\mathbb{P}_1$

In *machine learning*, we *have to estimate* $\mathbb{P}_0$ and $\mathbb{P}_1$ from data

Example in 1D: Spam

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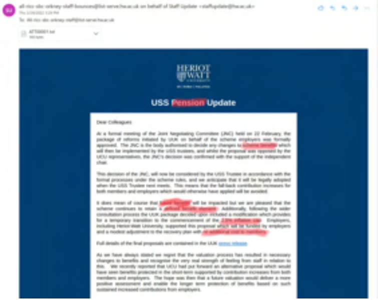

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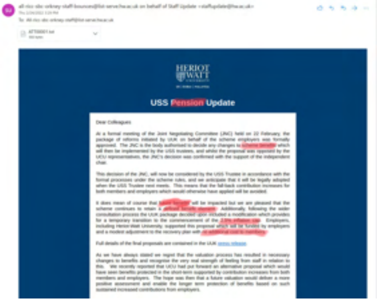

$X \in \mathbb{R}$: number of **spam words** in a message

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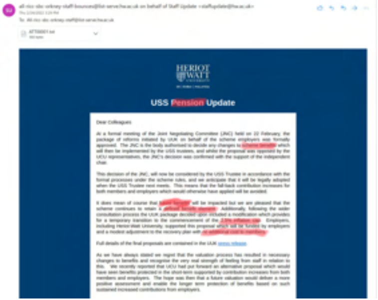

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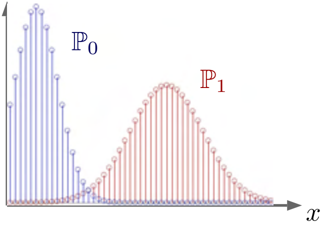

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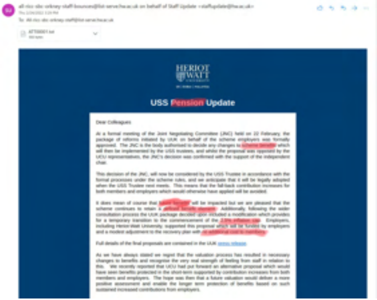

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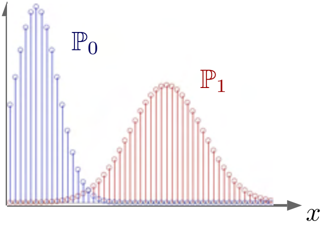
Null Hypothesis

H_0: message isn't spam



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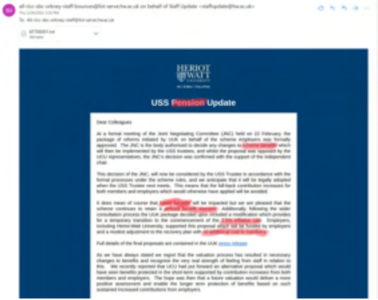

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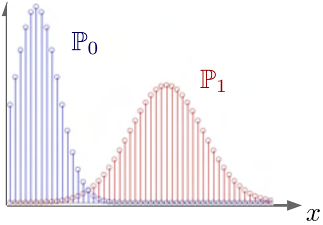

Null Hypothesis

H_0: message isn't spam

Alternative Hypothesis

H_1: message isn't spam

Signal vs Noise

Signal vs Noise

Consider a test for detecting:

if given email is spam

Signal vs Noise

Consider a test for detecting:

if given email is spam   presence of aircraft in radar

Signal vs Noise

Consider a test for detecting:

if given email is spam	presence of aircraft in radar
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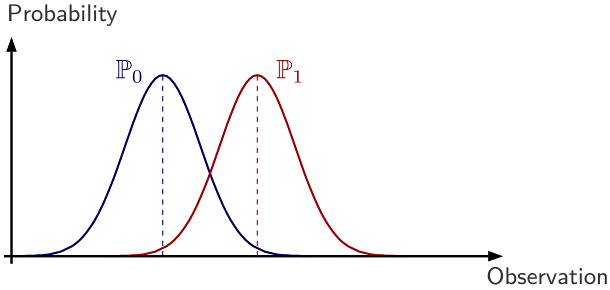

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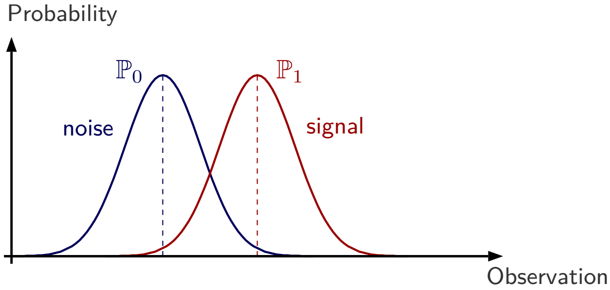
if defendant is guilty — presence of tumor in an image



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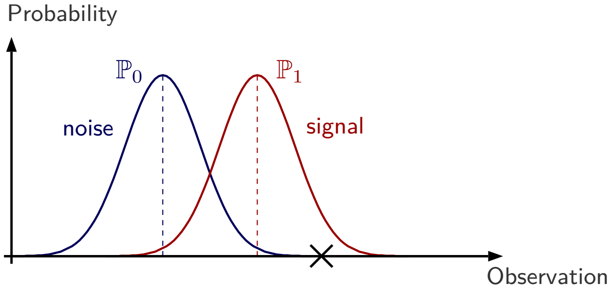

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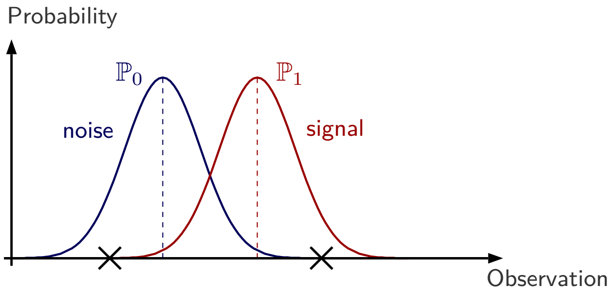

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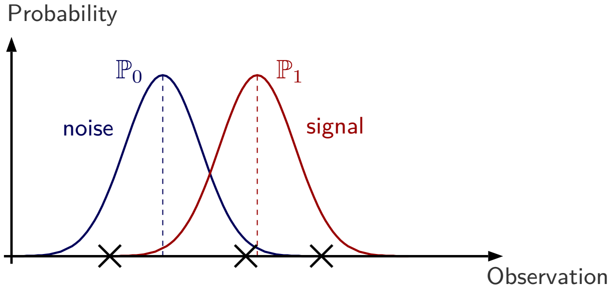

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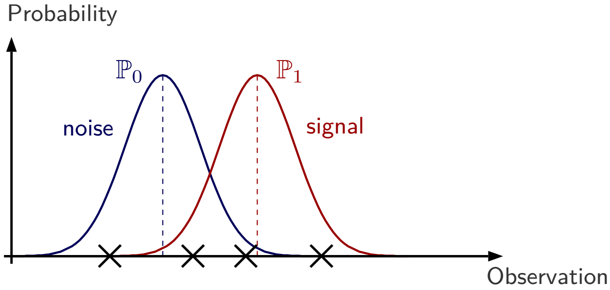

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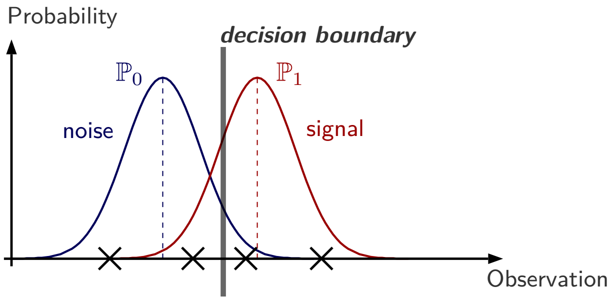

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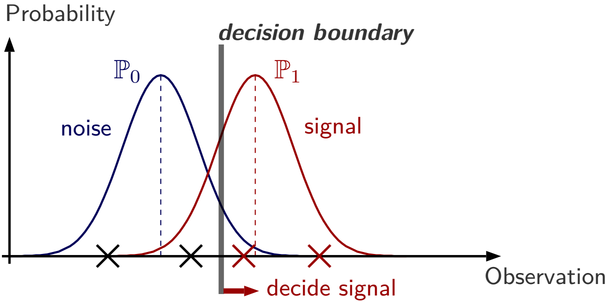

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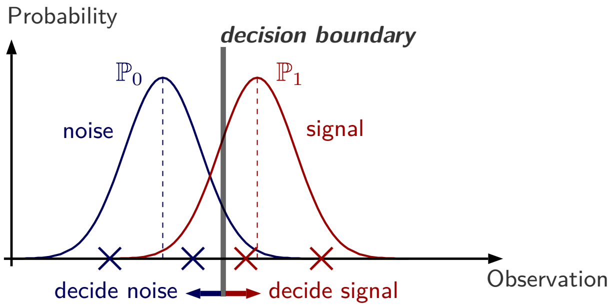

The Decision Tradeoff

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True hypothesis	Decide noise	Decide signal
noise		
signal		

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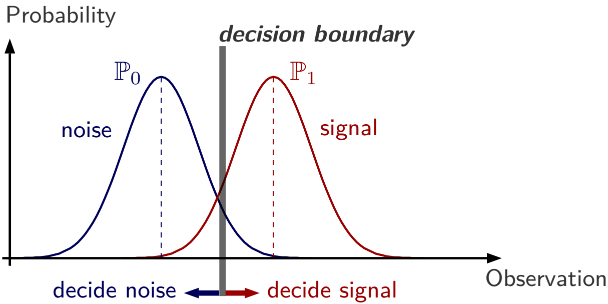
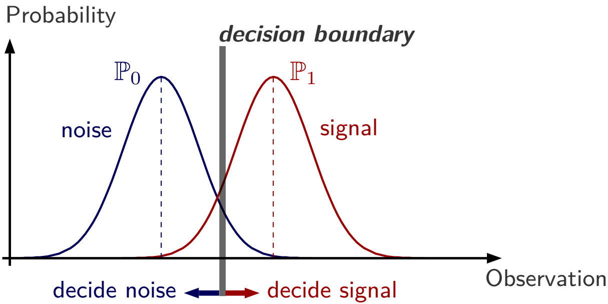


The Decision Tradeoff

True hypothesis	Decide noise	Decide signal
noise	✓	
signal		



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True hypothesis	Decide noise	Decide signal
noise	✓	
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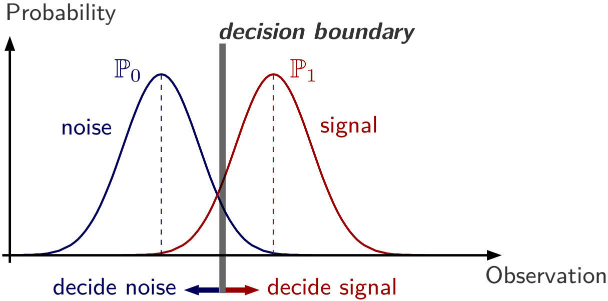

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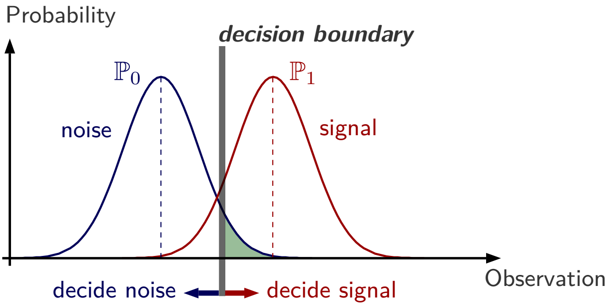
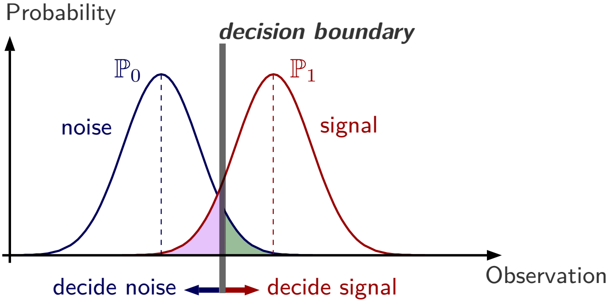
noise	✓	false alarm
signal		✓



The Decision Tradeoff

True hypothesis	Decide noise	Decide signal
noise	✓	false alarm
signal	missed detection	✓



Improving the Tradeoff

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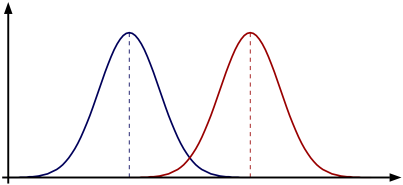
Larger effect size



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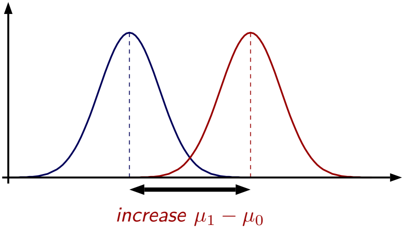

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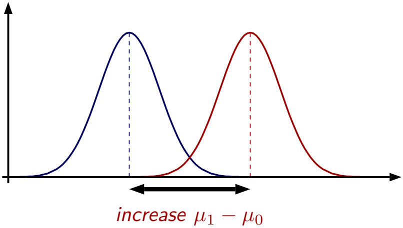
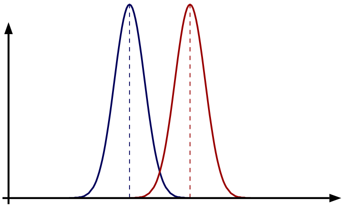


Better/more measurements



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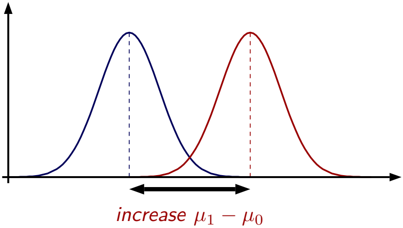

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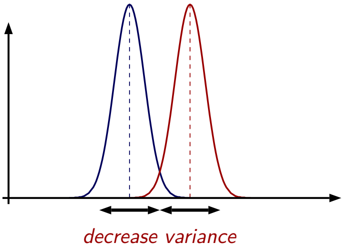
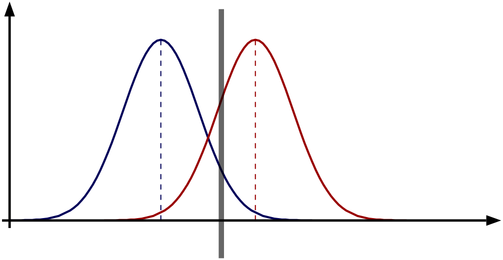
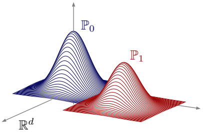


Where to place the decision boundary?



Decision and loss functions



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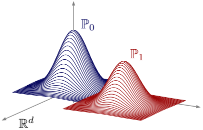

True label

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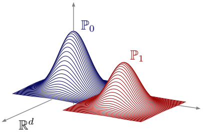

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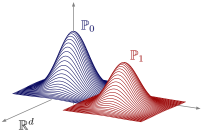

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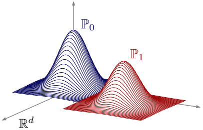
Decision and loss functions



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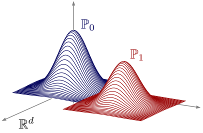

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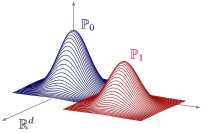

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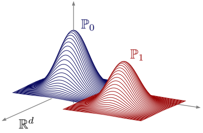

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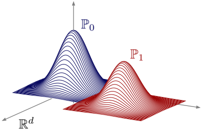

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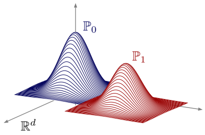

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Risk and Optimal Decision

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Given *decision function* $f : \mathbb{R}^d \to \{0, 1\}$ and *loss* $\ell : \{0, 1\}^2 \to \mathbb{R}$,

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Risk:

Risk and Optimal Decision

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$$R[f] := \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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where $\mathbb{E}_{\boldsymbol{X}Y}[\cdot]$ is the expectation with respect to $\boldsymbol{X}$ and Y

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Optimal decision problem:

Risk and Optimal Decision

Given *decision function* $f : \mathbb{R}^d \to \{0,1\}$ and *loss* $\ell : \{0,1\}^2 \to \mathbb{R}$,

Risk:

$$\boxed{R[f] := \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]}$$

where $\mathbb{E}_{\boldsymbol{X}Y}[\cdot]$ is the expectation with respect to $\boldsymbol{X}$ and Y

Optimal decision problem: Given ℓ, find f that minimizes the risk:

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. . . infinite-dimensional problem

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Recall that $f(\boldsymbol{X})$ and Y are binary

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\left[\ell\big(f(\boldsymbol{X}), Y\big)\right]$$

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Conditioning on $\boldsymbol{X}$,

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \;\; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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$$\mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}, Y)\big)\Big] = \mathbb{E}_{\boldsymbol{X}}\left[\mathbb{E}_{Y}\left[\ell\big(f(\boldsymbol{X}),\, Y\big) \,\Big|\, \boldsymbol{X}\right]\right]$$

$$\underset{f:\mathbb{R}^d\to\{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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Conditioning on $\boldsymbol{X}$,

$$\begin{aligned}\mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X},Y)\big)\Big] &= \mathbb{E}_{\boldsymbol{X}}\left[\mathbb{E}_Y\left[\ell\big(f(\boldsymbol{X}),\, Y\big)\,\Big|\, \boldsymbol{X}\right]\right]\\ &= \int_{\mathbb{R}^d} \mathbb{E}_Y\left[\ell\big(f(\boldsymbol{X}),\, Y\big)\,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right] f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

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\end{aligned}$$

If $f(\boldsymbol{x}) = 0$,

$$\underset{f:\mathbb{R}^d\to\{0,1\}}{\text{minimize}}\;\mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}),Y\big)\Big]$$

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If $f(\boldsymbol{x})=0$,

$$\mathbb{E}_Y\left[\ell\big(0,\,Y\big)\,\Big|\,\boldsymbol{X}=\boldsymbol{x}\right] = \ell(0,0)\,\mathbb{P}\big(Y=0\,|\,\boldsymbol{X}=\boldsymbol{x}\big)+\ell(0,1)\,\mathbb{P}\big(Y=1\,|\,\boldsymbol{X}=\boldsymbol{x}\big)$$

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$$\mathbb{E}_Y\Big[\ell\big(0,\, Y\big)\,\Big|\,\boldsymbol{X}=\boldsymbol{x}\Big] = \ell(0,0)\,\mathbb{P}\big(Y=0\,|\,\boldsymbol{X}=\boldsymbol{x}\big) + \ell(0,1)\,\mathbb{P}\big(Y=1\,|\,\boldsymbol{X}=\boldsymbol{x}\big)$$

If $f(\boldsymbol{x}) = 1$,

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$$\mathbb{E}_Y\Big[\ell\big(0,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\Big] = \ell(0,0)\,\mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) + \ell(0,1)\,\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

If $f(\boldsymbol{x}) = 1$,

$$\mathbb{E}_Y\Big[\ell\big(1,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\Big] = \ell(1,0)\,\mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) + \ell(1,1)\,\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

Optimal decision

Optimal decision

$$f(\boldsymbol{x}) = 0 \qquad \text{if} \qquad \mathbb{E}_Y\left[\ell\big(0,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1,\, Y\big) \,\Big|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Rearranging, the optimal decision is

$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(Y = 1 \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)}\, \mathbb{P}\big(Y = 0 \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}$$

Optimal decision

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\begin{aligned}
f(\boldsymbol{x}) &= 0 \qquad \text{if} \qquad \mathbb{E}_Y\left[\ell\big(0,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] \\
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Rearranging, the optimal decision is

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f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overset{H_1}{Y = 1} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)}\, \mathbb{P}\big(Y = 0 \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}
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$$
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f(\boldsymbol{x}) = 0 \qquad &\text{if} \qquad \mathbb{E}_Y\left[\ell\big(0,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] \\
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f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overset{H_1}{Y = 1} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)}\, \mathbb{P}\big(\overset{H_0}{Y = 0} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}
$$

Bayes rule

Optimal decision

$$f(\boldsymbol{x}) = 0 \quad \text{if} \quad \mathbb{E}_Y\left[\ell\big(0,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1,\, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Rearranging, the optimal decision is

$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overbrace{Y = 1}^{H_1} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)}\, \mathbb{P}\big(\overbrace{Y = 0}^{H_0} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}$$

Bayes rule

$$\frac{f_{\boldsymbol{X} \mid H_1}(\boldsymbol{x} \mid H_1)\, \mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

Optimal decision

$$
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\end{aligned}
$$

Rearranging, the optimal decision is

$$
f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overset{H_1}{Y = 1}\,\big|\,\boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)}\,\mathbb{P}\big(\overset{H_0}{Y = 0}\,\big|\,\boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}
$$

Bayes rule

$$
\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x}\,|\,H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad\qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x}\,|\,H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}
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Bayes rule

$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \,|\, H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad\qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \,|\, H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

$$f(\boldsymbol{x}) = 1 \quad \text{if} \quad \frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \,|\, H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \,|\, H_0)} \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}$$

Optimal decision

$$f(\boldsymbol{x}) = 0 \quad \text{if} \quad \mathbb{E}_Y\left[\ell\big(0, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell\big(1, Y\big) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Bayes rule

$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \,|\, H_1)\, \mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \,|\, H_0)\, \mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

likelihood ratio test

$$f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \,|\, H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \,|\, H_0)} \geq \frac{\ell(0,0) - \ell(1,0)}{\ell(1,1) - \ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}$$

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$$
\begin{aligned}
f(\boldsymbol{x}) = 0 \quad &\text{if} \quad \mathbb{E}_Y\left[\ell(0,\,Y)\,\middle|\,\boldsymbol{X}=\boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell(1,\,Y)\,\middle|\,\boldsymbol{X}=\boldsymbol{x}\right] \\
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\end{aligned}
$$

Rearranging, the optimal decision is

$$
f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \underbrace{\mathbb{P}(\overset{H_1}{Y=1}\mid \boldsymbol{X}=\boldsymbol{x})}_{\dfrac{f_{\boldsymbol{X}\mid H_1}(\boldsymbol{x}\mid H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})}} \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)}\,\underbrace{\mathbb{P}(\overset{H_0}{Y=0}\mid \boldsymbol{X}=\boldsymbol{x})}_{\dfrac{f_{\boldsymbol{X}\mid H_0}(\boldsymbol{x}\mid H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}} \\ 0 & \text{otherwise} \end{cases}
$$

Bayes rule

likelihood ratio test

$$
f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\underbrace{\frac{f_{\boldsymbol{X}\mid H_1}(\boldsymbol{x}\mid H_1)}{f_{\boldsymbol{X}\mid H_0}(\boldsymbol{x}\mid H_0)}}_{\mathcal{L}(\boldsymbol{x})} \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}
$$

Optimal decision

$$f(\boldsymbol{x}) = 0 \quad \text{if} \quad \mathbb{E}_Y\left[\ell(0, Y) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right] < \mathbb{E}_Y\left[\ell(1, Y) \,\middle|\, \boldsymbol{X} = \boldsymbol{x}\right]$$

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Rearranging, the optimal decision is

$$f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\big(\overset{H_1}{Y = 1} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)} \, \mathbb{P}\big(\overset{H_0}{Y = 0} \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\ 0 & \text{otherwise} \end{cases}$$

Bayes rule

$$\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)\,\mathbb{P}(H_1)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \frac{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)\,\mathbb{P}(H_0)}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

likelihood ratio test

$$f(\boldsymbol{x}) = 1 \quad \text{if} \quad \boxed{\frac{f_{\boldsymbol{X}|H_1}(\boldsymbol{x} \mid H_1)}{f_{\boldsymbol{X}|H_0}(\boldsymbol{x} \mid H_0)} \geq \frac{\ell(0,0)-\ell(1,0)}{\ell(1,1)-\ell(0,1)} \cdot \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)}}$$

$\mathcal{L}(\boldsymbol{x})$: likelihood ratio

Likelihood ratio test

$$\underset{f:\mathbb{R}^d\to\{0,1\}}{\text{minimize}} \; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}),Y\big)\Big]$$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x}) \geq \eta\}}(\boldsymbol{x})$$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \;\; \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

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$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x}) \geq \eta\}}(\boldsymbol{x})$$

- Indicator function of set $\mathcal{S}$: $\quad \mathbb{1}_{\mathcal{S}}(s) = \begin{cases} 1 & , \text{ if } s \in \mathcal{S} \\ 0 & , \text{ if } s \notin \mathcal{S} \end{cases}$

Likelihood ratio test

$$\underset{f:\mathbb{R}^d \to \{0,1\}}{\text{minimize}} \ \mathbb{E}_{\boldsymbol{X}Y}\Big[\ell\big(f(\boldsymbol{X}), Y\big)\Big]$$

The decision that minimizes the risk in a binary hypothesis test is

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- *Likelihood ratio:* $\quad \mathcal{L}(\boldsymbol{x}) = \dfrac{f_{\boldsymbol{X}|H_1}\big(\boldsymbol{x} \,\big|\, H_1\big)}{f_{\boldsymbol{X}|H_0}\big(\boldsymbol{x} \,\big|\, H_0\big)}$

Likelihood ratio test

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The decision that minimizes the risk in a binary hypothesis test is

$$f(\boldsymbol{x}) = \mathbb{1}_{\{\mathcal{L}(\boldsymbol{x})\geq\eta\}}(\boldsymbol{x})$$

- Indicator function of set $\mathcal{S}$: $\mathbb{1}_{\mathcal{S}}(s) = \begin{cases} 1 & , \text{ if } s\in\mathcal{S} \\ 0 & , \text{ if } s\notin\mathcal{S} \end{cases}$
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$H_0 \,:\, X = W$

$H_1 \,:\, X = c + W$

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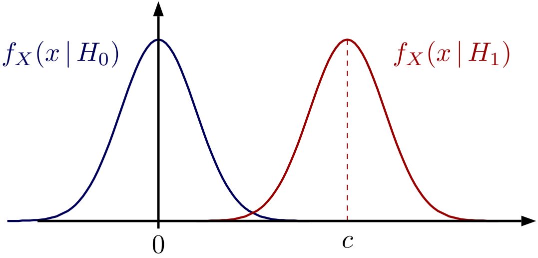

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Compute the decision threshold

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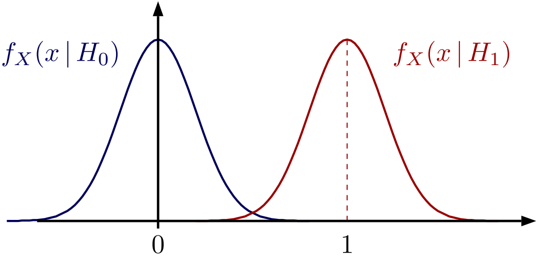

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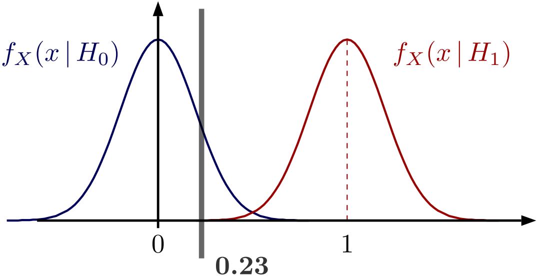

Particular cases

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When class $Y \in \{0,1\}$ is viewed as a parameter of $\mathbb{P}_{\boldsymbol{X}Y}$ to estimate,

- Maximum a posteriori (MAP)
- Maximum likelihood (ML)

can be seen as *likelihood ratio tests*

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$$\iff \quad f_{\boldsymbol{X}|H_1}(x \,|\, H_1) \cdot \mathbb{P}(H_1) \geq f_{\boldsymbol{X}|H_0}(x \,|\, H_0) \cdot \mathbb{P}(H_0)$$

$$\iff \quad \mathcal{L}(\boldsymbol{x}) = \frac{f_{\boldsymbol{X}|H_1}(x \,|\, H_1)}{f_{\boldsymbol{X}|H_0}(x \,|\, H_0)} \geq \frac{\mathbb{P}(H_0)}{\mathbb{P}(H_1)} = \eta$$

So, select $f(\boldsymbol{x}) = 1$ if $\mathbb{P}\big(Y = 1 \,|\, \boldsymbol{X} = \boldsymbol{x}\big) \geq \mathbb{P}\big(Y = 0 \,|\, \boldsymbol{X} = \boldsymbol{x}\big)$

That is, $$f(\boldsymbol{x}) = \arg\max_i \underbrace{\mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big)}_{\textit{posterior}}$$

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Recall that MAP rule minimizes probability of incorrect decision:

$$\mathbb{P}\big(\text{error}\big) = \mathbb{P}\big(f(\boldsymbol{X}) = 1\,,\ H_0\big) + \mathbb{P}\big(f(\boldsymbol{X}) = 0\,,\ H_1\big)$$

Maximum Likelihood (ML)

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$$f(\boldsymbol{x}) \;=\; \arg\max_i \;\; \mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big)$$

$$=\; \arg\max_i \;\; \frac{f_{\boldsymbol{X}|Y}(\boldsymbol{x} \,|\, Y = i) \cdot \mathbb{P}(Y = i)}{f_{\boldsymbol{X}}(\boldsymbol{x})} \qquad \Big(\text{Bayes rule}\Big)$$

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$$
\begin{aligned}
f(\boldsymbol{x}) &= \arg\max_i \ \mathbb{P}\big(Y = i \,\big|\, \boldsymbol{X} = \boldsymbol{x}\big) \\
&= \arg\max_i \ \frac{f_{\boldsymbol{X}|Y}(\boldsymbol{x} \,|\, Y = i) \cdot \mathbb{P}(Y = i)}{f_{\boldsymbol{X}}(\boldsymbol{x})} && \Big(\text{Bayes rule}\Big) \\
&= \arg\max_i \ f_{\boldsymbol{X}|Y}\Big(\boldsymbol{x} \,\Big|\, Y = i\Big) && \left(\mathbb{P}(Y = i) = \frac{1}{2}\right)
\end{aligned}
$$

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This corresponds to a likelihood ratio test with $\eta = 1$

Types of errors and successes

Types of errors and successes

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
H_1 is true		

Types of errors and successes

True Positive Rate (TPR)

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
H_1 is true		

Types of errors and successes

True Positive Rate (TPR)
power, sensitivity, recall

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
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True Positive Rate (TPR)
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H_0 is true		
H_1 is true		TPR

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$$\text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$$

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
H_1 is true		TPR

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True Positive Rate (TPR)
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$$\text{TPR} = \mathbb{P}\left(f(\boldsymbol{X}) = 1 \,\middle|\, H_1\right)$$

False Positive Rate (FPR)
type I error, false alarm

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		
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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
H_1 is true		TPR

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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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type I error, false alarm

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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type I error, false alarm

True Negative Rate (TNR)

specificity

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true		FPR
H_1 is true		TPR

Types of errors and successes

True Positive Rate (TPR) $\qquad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
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True Positive Rate (TPR) $\quad \mathrm{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
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True Negative Rate (TNR) $\quad \text{TNR} = \mathbb{P}\big(f(\boldsymbol{X}) = 0 \,\big|\, H_0\big)$

specificity

False Negative Rate (FNR)

type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
H_1 is true		TPR

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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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type I error, false alarm

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR
H_1 is true	FNR	TPR

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True Positive Rate (TPR) $\quad \text{TPR} = \mathbb{P}\big(f(\boldsymbol{X}) = 1 \,\big|\, H_1\big)$

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True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
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type II error, missed detection

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
H_1 is true	FNR	TPR

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True Positive Rate (TPR)
power, sensitivity, recall

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type I error, false alarm

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specificity

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type II error, missed detection

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Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
H_1 is true	FNR $\boldsymbol{\beta}$	TPR

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Precision: $\mathbb{P}\big(H_1 \,\big|\, f(\boldsymbol{X}) = 1\big)$

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Precision: $$\mathbb{P}\big(H_1 \,\big|\, f(\boldsymbol{X}) = 1\big) = \frac{\mathsf{TPR} \cdot \mathbb{P}(H_1)}{\mathsf{TPR} \cdot \mathbb{P}(H_1) + \mathsf{FPR} \cdot \mathbb{P}(H_0)}$$

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$\boldsymbol{F_1}$**-score:** harmonic mean between precision and recall (TPR):

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***F_1*-score:** harmonic mean between precision and recall (TPR):

$$F_1 = \frac{2}{\frac{1}{\mathbb{P}(H_1 \mid f(\boldsymbol{X})=1)} + \frac{1}{\mathsf{TPR}}}$$

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
H_0 is true	TNR	FPR $\boldsymbol{\alpha}$
H_1 is true	FNR $\boldsymbol{\beta}$	TPR

Table of probabilities

True hypothesis	$f(\boldsymbol{X}) = 0$	$f(\boldsymbol{X}) = 1$
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α and β are in *conflict:*

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α and β are in *conflict:*

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It turns out that likelihood ratio tests are *Pareto optimal*

Neyman-Pearson Lemma

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Theorem

Let $f_{\mathsf{LRT}}(\boldsymbol{x})$ be a likelihood ratio decision rule

Neyman-Pearson Lemma

Theorem

Let $f_{\mathsf{LRT}}(\boldsymbol{x})$ be a likelihood ratio decision rule with FPR and FNR

$$\alpha_{\mathsf{LRT}} = \mathbb{P}\Big(f_{\mathsf{LRT}}(\boldsymbol{x}) = 1 \,\Big|\, H_0\Big) \qquad \beta_{\mathsf{LRT}} = \mathbb{P}\Big(f_{\mathsf{LRT}}(\boldsymbol{x}) = 0 \,\Big|\, H_1\Big)$$

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Let $f(\boldsymbol{x})$ be another (deterministic or probabilistic) decision rule with

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$$\alpha \le \alpha_{\mathsf{LRT}} \qquad \Longrightarrow \qquad \beta \ge \beta_{\mathsf{LRT}}$$

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Theorem

Let $f_{\mathsf{LRT}}(\boldsymbol{x})$ be a likelihood ratio decision rule with FPR and FNR

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And the same relations hold with strict inequalities $(<, >)$

Proof

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$$\mathbb{P}\Big(\mathsf{error}_{\mathsf{MAP}}\Big)$$

$$= \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\ H_0\Big) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\ H_1\Big)$$

Proof

- $f_{\mathsf{MAP}}(\boldsymbol{x})$ minimizes probability of error over all rules (det. and prob.):

$$
\begin{aligned}
&\mathbb{P}\Big(\mathsf{error}_{\mathsf{MAP}}\Big) \\
&= \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\ H_0\Big) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1,\ H_1\Big) \\
&= \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1 \,\Big|\, H_0\Big)\ \mathbb{P}(H_0) + \mathbb{P}\Big(f_{\mathsf{MAP}}(\boldsymbol{X}) = 1 \,\Big|\, H_1\Big)\ \mathbb{P}(H_1)
\end{aligned}
$$

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  This implies $f_{\text{LRT}}(\boldsymbol{x}) = f_{\text{MAP}}(\boldsymbol{x})$

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$\square$

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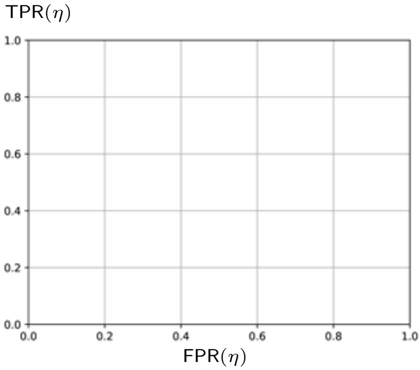

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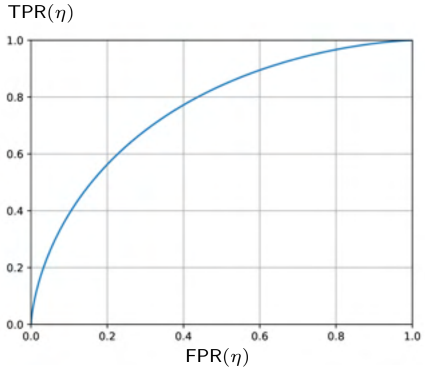

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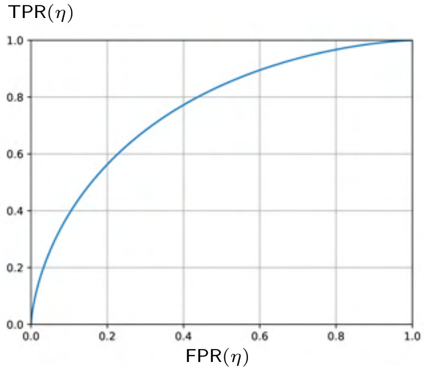

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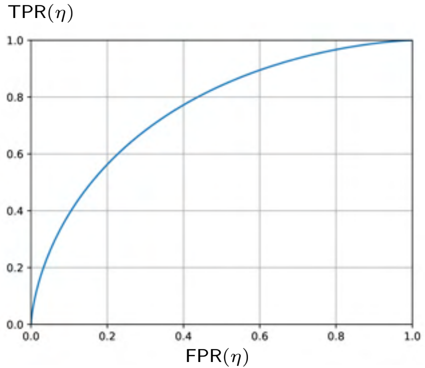

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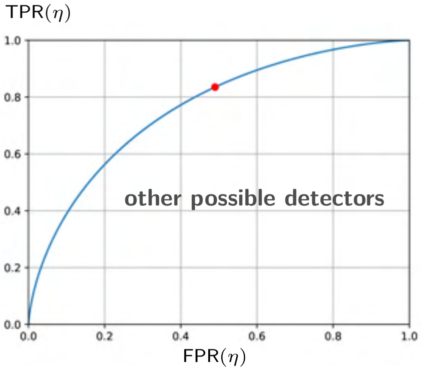

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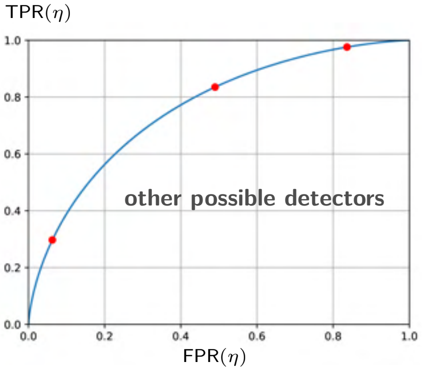

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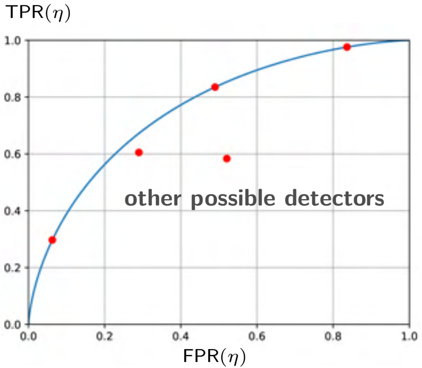

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Example

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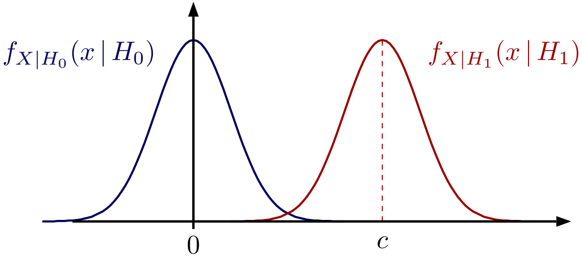

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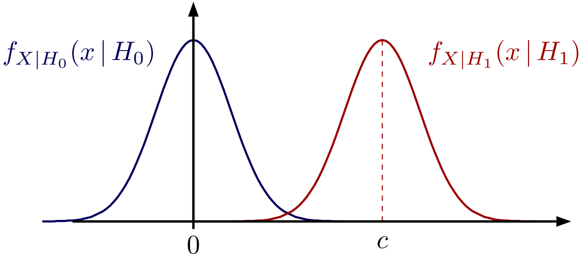

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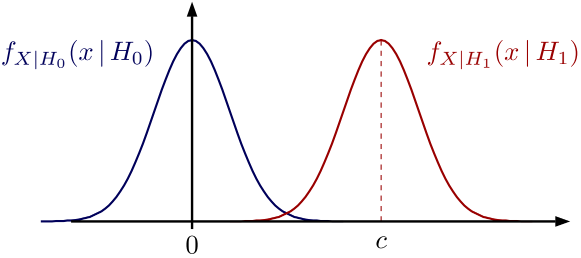

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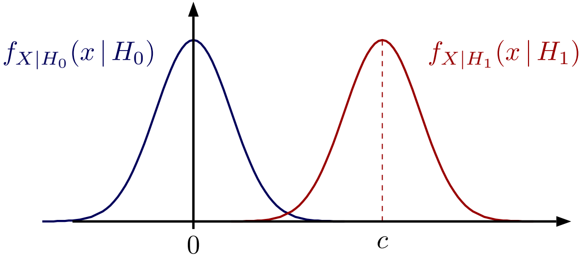

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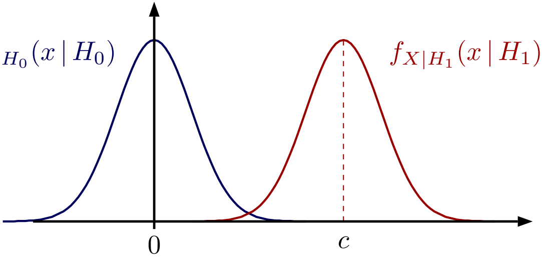

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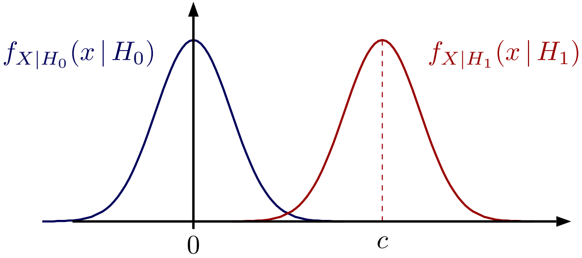

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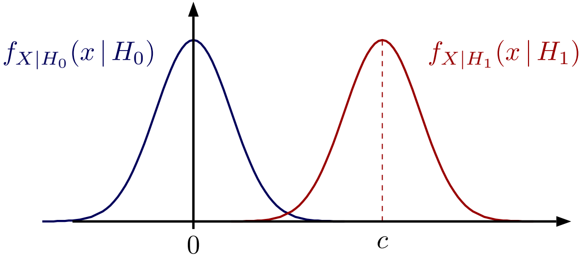

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&= Q\left(\frac{\gamma - c}{\sigma}\right) \\
&= Q\left(\frac{\log \eta}{c/\sigma} - \frac{c/\sigma}{2}\right)
\end{aligned}$$

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&= \int_{\frac{\gamma - c}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, \mathrm{d}z \\
&= Q\left(\frac{\gamma - c}{\sigma}\right) \\
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\end{aligned}$$

$\left[Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{z^2}{2}} \, \mathrm{d}z\right]$

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$x \geq \frac{c}{2} + \frac{\sigma^2}{c} \log \eta := \gamma$

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ROC curve for different values of SNR

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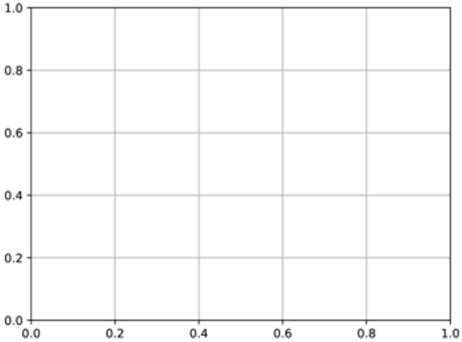

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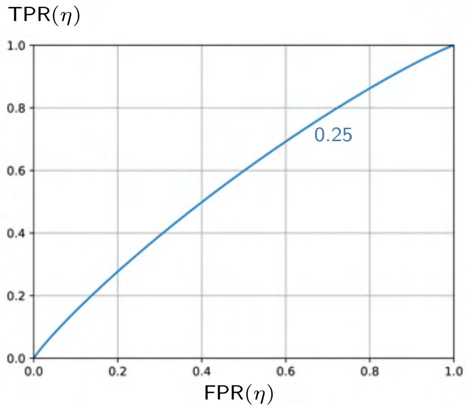

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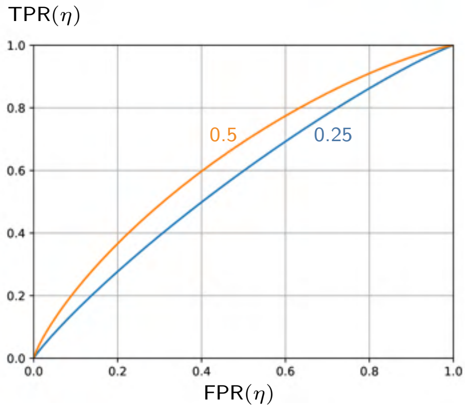

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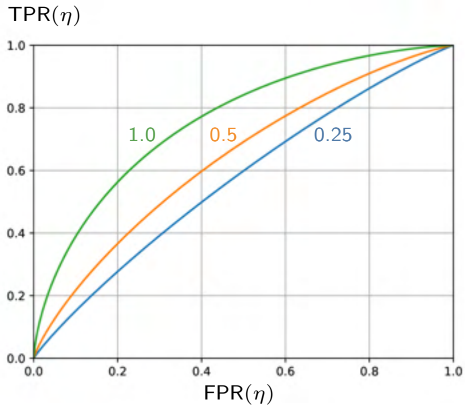

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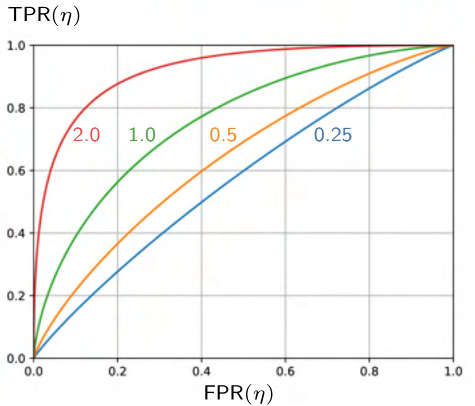

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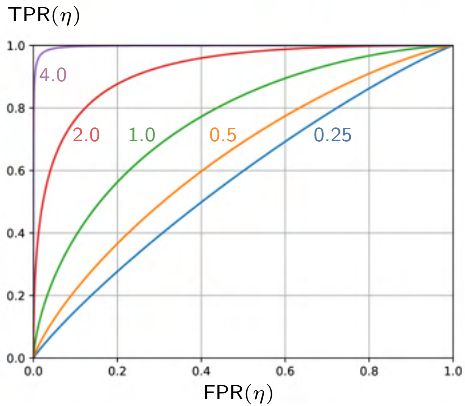

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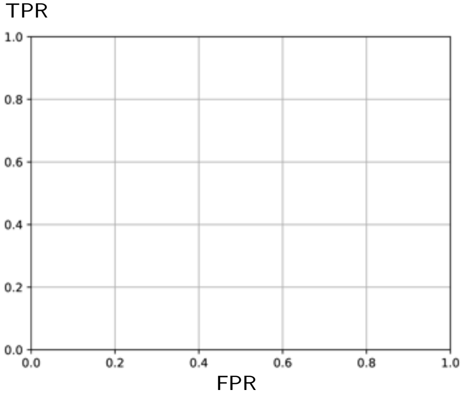

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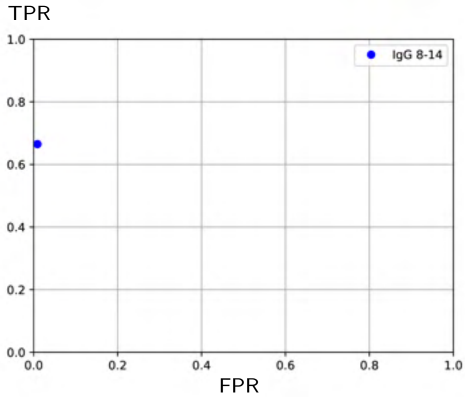

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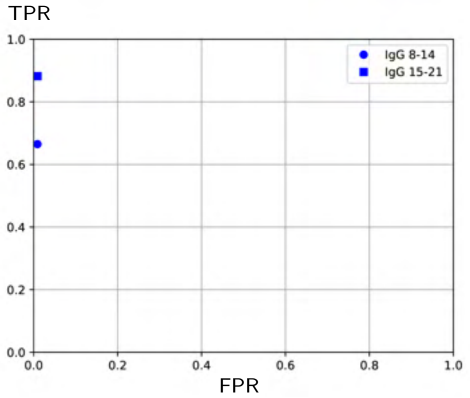

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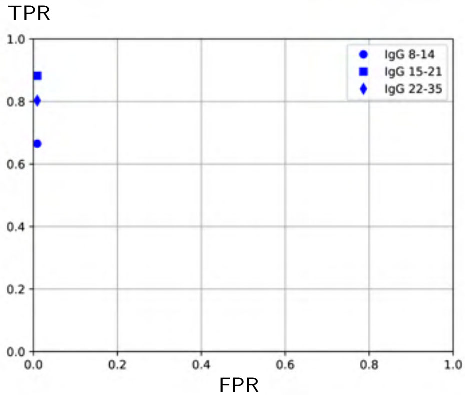

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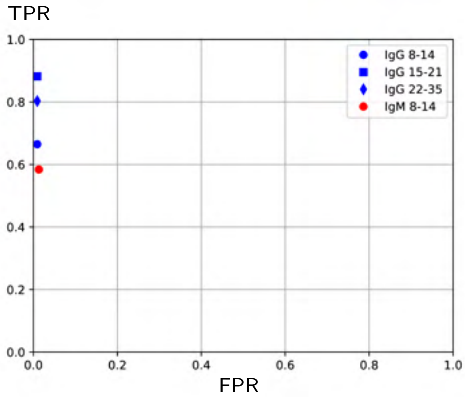

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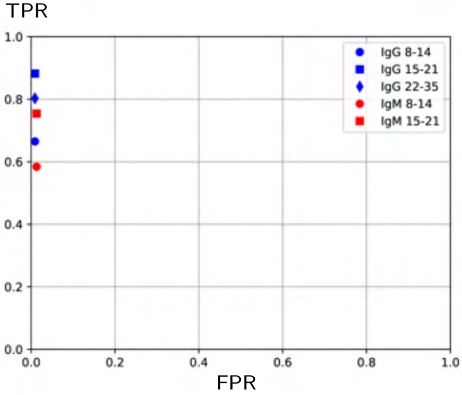

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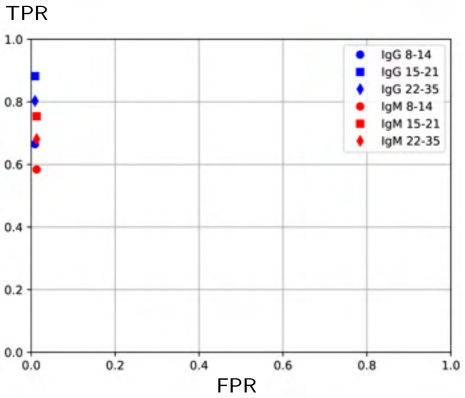

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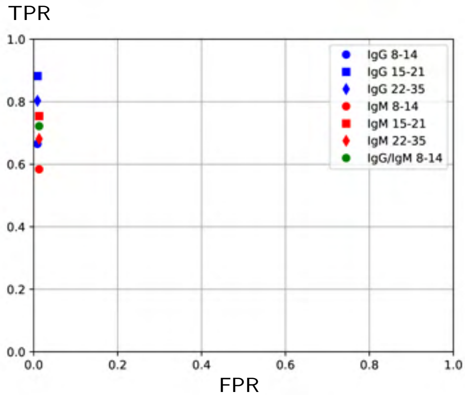

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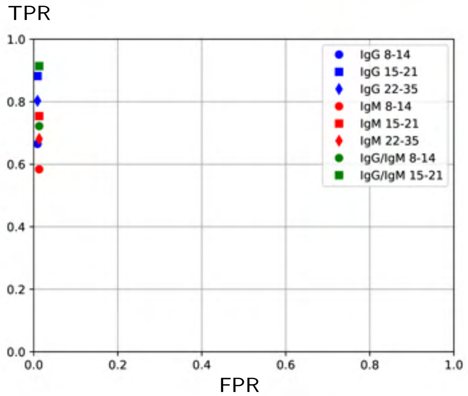

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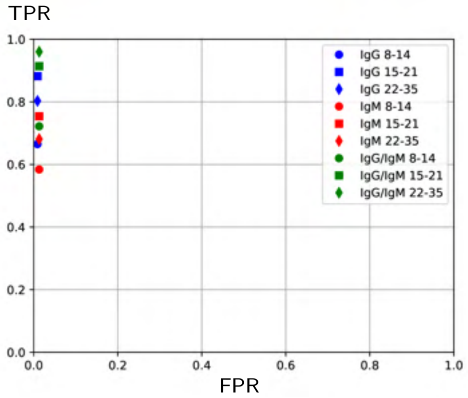

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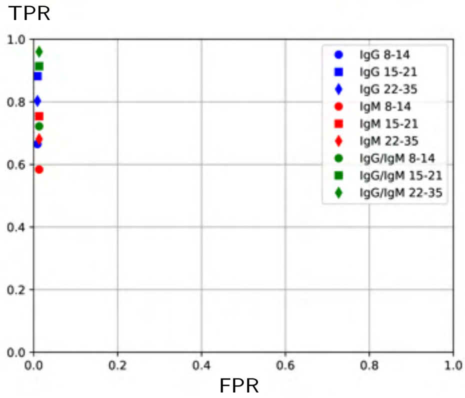

Deeks et al, **Antibody tests for identification of current and past infection with SARSCoV2**, Cochrane Database of Systematic Reviews, Issue 6, 2020

Looking Ahead: Empirical Risk Minimization

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Assumed known $f_{\boldsymbol{X}\,|\,H_i}(\boldsymbol{x}\,|\,H_i)$ and $\mathbb{P}(H_i)$

What if they are unknown?

Looking Ahead: Empirical Risk Minimization

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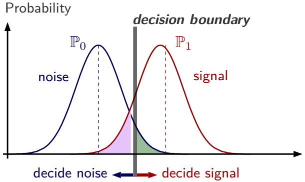
Assumption: we observe T samples $\{(\boldsymbol{x}_t,\, y_t)\}_{t=1}^{T}$

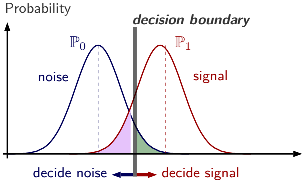
Conclusions

Conclusions



Optimal decision problems (binary case)

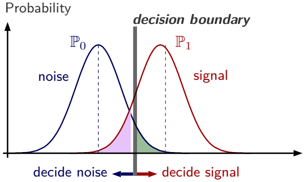
Conclusions



Optimal decision problems (binary case)

Likelihood ratio tests (LRT)

Conclusions



Optimal decision problems (binary case)

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MAP and ML as particular cases

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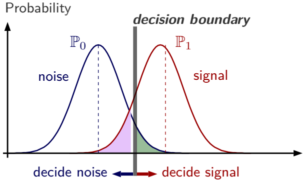

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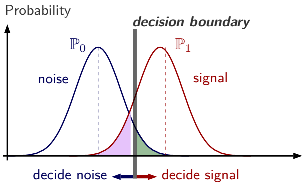

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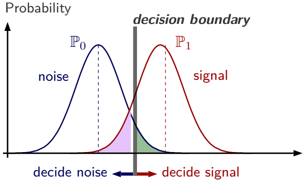

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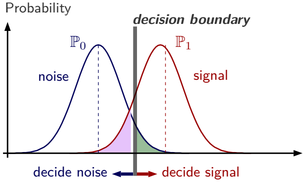

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ROC curves and properties

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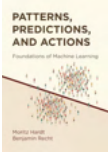

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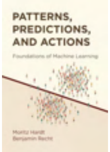

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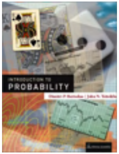

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