A LOW-COMPLEXITY SUB-NYQUIST SAMPLING SYSTEM
FOR WIDEBAND RADAR ESM RECEIVERS

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ABSTRACT

The problem of efficient sampling of wideband Radar signals for Electronic Support Measures (ESM) is investigated in this paper. Wideband radio frequency sampling generally needs a sampling rate at least twice the maximum frequency of the signal, *i.e.* Nyquist rate, which is generally very high. However, when the signal is highly structured, like wideband Radar signals, we can use the fact that signals do not occupy the whole spectrum and instead, there exists a parsimonious structure in the time-frequency domain. Here, we use this fact and introduce a novel low complexity sampling system, which has a recovery guarantee, assuming that received RF signals follow a particular structure. The proposed technique is inspired by the compressive sampling of sparse signals and it uses a multi-coset sampling setting, however it does not involve a computationally expensive reconstruction step. We call this here Low-Complexity Multi-Coset (LoCoMC) sampling technique. Simulation results, show that the proposed sub-Nyquist sampling technique works well in simulated ES scenarios.

Index Terms— Sub-Nyquist Sampling, Compressive Sampling, Electronic Surveillance, Electronic Support Measures, Wideband Radar

1. INTRODUCTION

In electronic surveillance, we need to monitor a wide frequency band, where the Radar and communication signals occupy different bands. It has been considered for many years and various solutions have already been proposed. Early solutions used instantaneous frequency measurements (IFM) to detect and categorize RF signals, see for example [1]. However, such systems have limited sensitivity and cannot sort multiple signals simultaneously. In more recent years, digital receivers have been preferred due to their ability to process a wider range of signals including Frequency Modulated Continuous Wave (FMCW) radar signals and other Low Probability of Intercept (LPI) radar signals. However, size weight and power (SWAP) requirements impose limitations on the sampling rates and hence the bandwidths that are viable for such digital receivers.

An alternative approach to wideband sampling, which has received much interest recently, is to use a sub-Nyquist sampling strategy that allows subsequent reconstruction of the signal under certain assumptions, such as sparsity. The earliest such system was proposed by Feng and Bresler [2] and exploited a multi-coset (MC)

sampling strategy: a parallel bank of sub-Nyquist sampling channels, each with their own unique delay. Signal reconstruction is possible as long as the number of active subbands is less that the number of multi-coset channels. Identification of the active subbands is achieved using a variant of the MUSIC algorithm [3] which involves a computationally expensive eigenvalue decomposition and requires an accumulation of sufficient samples to calculate an accurate cross channel covariance matrix. Multi-coset sampling was revisited by Mishali and Eldar in [4] where variations of standard compressed sensing algorithms were proposed for signal reconstruction. Other sub-Nyquist sampling strategies have also been proposed based on the ideas of compressed sensing [5]. For example, Random Demodulation (RD) sampling strategies using spread spectrum techniques similar to those in telecommunications have recently been proposed [6, 7]. However, as in [4], the reconstruction technique proposed for all these systems are *iterative* and they are thus computationally expensive, which does not allow us to use them for large scale problems like ESM, see [8] and reference therein for the setting of digital ESM problem. MacKerron *et al.* in [9], proposed a technique to break the large scale reconstruction problem to a series of small size problems. Such a technique will be more efficient, if we have some techniques to solve small problems quickly. If we use greedy algorithms for this purpose, the process is still iterative.

We here propose an implementation of a wideband sub-Nyquist rate receiver, which is able to acquire and efficiently reconstruct signals that have an approximate disjoint aliased TF support, using a very simple pipelined process. The new signal model can accommodate to a wide range of TF-sparse signals. ESM is the application that we have focused here and leave the generalisation to other applications for a future work.
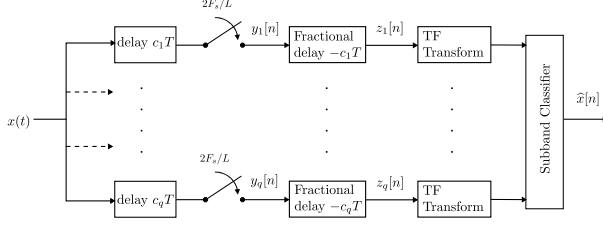
2. SUB-NYQUIST SAMPLING SYSTEM PROPOSAL

We initially introduce our sampling framework and mathematically formulate its operation here. The sampling structure is composed of two separate parts: the analog circuit, *i.e.* digitiser, and the digital processing unit. The digitiser consists of a bank of parallel delayed signals, with distinguished delays, each sampled with a fixed rate lower than Nyquist. Signals of different channels are called cosets and the whole system is called a multi-coset sampling scheme [2]. Unlike the work of Feng and Bresler [2] or Mishali and Eldar [4], we propose a receiver which can even be implemented with *as few as two multi-coset channels*, while increasing the number of channels, increases the robustness of the sampling technique to the noise.

The digital component of the receiver is composed of a digital fractional delay (DFD) filter and a Time Frequency (TF) transform per channel followed by a joint detection and de-aliasing step. The

Fig. 1. The proposed sub-Nyquist sampling system (LoCoMC).

whole process can be pipelined and is non-iterative. It is therefore ideally suitable for a low SWAP implementation. A complete system diagram is shown in Figure 1 and consists of the following elements. The input signal $x(t)$ is sampled using a bank of sub-Nyquist sample and hold devices, each sampling at L times lower than the Nyquist rate $1/T$. Prior to the sample and hold, each channel is delayed by a unique time delay of $c_i T$ seconds. Following track and hold, the signal is digitized using an Analog to Digital Converter (ADC) for subsequent digital processing and analysis.

2.1. Mathematical Formulation

Consider an input signal, $x(t)$. Following [2], examination of the ith channel shows that the input signal, $x(t)$, sampled at a rate of $1/LT$ can be written as $y_i[n] = x((nL+c_i)T)$. For simplicity of notation, we assume that the support of the Fourier transform of $x(t)$, denoted by $X(\omega)$, is *essentially* band-limited to $[0, 2\pi/T]$. Plainly speaking, essentially band-limited functions have negligible out of the band energy. A unit energy signal is thus essentially band-limited, if the out of the band energy is smaller than $1 - \epsilon$, for a small epsilon, see [10] for more information.

While we chose here $[0, 2\pi/T]$ as the essential band limit of the signal, generalizing to other frequency supports is straight forward. We can therefore write the Discrete Time Fourier Transform (DTFT) of $y_i[n]$ as:

$$Y_i(e^{j\omega LT}) = \frac{1}{LT} \sum_{l=0}^{L-1} e^{-jc_i T(\omega - 2\pi l/LT)} X(\omega - 2\pi l/LT) \quad (1)$$

Applying a reversed digital fractional delay of $-c_i T$ seconds to $y_i[n]$ yields $z_i[n]$. The DTFT of $z_i[n]$ is therefore:

$$Z_i(e^{j\omega LT}) = \frac{1}{LT} \sum_{l=0}^{L-1} e^{j2\pi l c_i/L} X(\omega - 2\pi l/LT) \quad (2)$$

Therefore, the output $z_i[n]$ is the superposition of L subband components of $x(t)$ multiplied by a phase shift that depends on the aliased band number l and the channel delay, c_i [2].

2.1.1. Time Frequency Representation

We are interested in signals that are in some sense sparse in a Gabor-based Time Frequency (TF) representation. Specifically, consider a TF atom of the form:

$$g_{m,k}(t) = g(t - m\tau_0)e^{j2\pi k\xi_0 t} \quad (3)$$

where $g(t)$ defines the window function [11], which is assumed to be normalized, $\|g\|_2 = 1$, *essentially* band-limited to $\omega \in [0, 2\pi/LT)$ and have its temporal support in the interval $0 \leq t < LNT$. Such

a class of TF transforms, includes a wide range of useful transforms for spectral analysis, *e.g.* STFT and the Chirplet transform.

The values τ_0 and ξ_0 define the discrete TF lattice, $(\tau, \xi) \in \{(m\tau_0, k\xi_0)|(m, k) \in \mathbb{Z}^2\}$ of the TF representation. For convenience we will restrict our attention to TF lattices such that $\tau_0 = MLT$ for some integer M and $\xi_0 = 1/KLT$ for some integer K. The frame coefficients of $x(t)$ can then be calculated as:

$$s_{m,k} := \langle x(t), g_{m,k}(t) \rangle$$
$$= \int_{-\infty}^{\infty} x(t)g^*(t - m\tau_0)e^{-j2\pi kt/LKT}dt \qquad (4)$$
$$= \frac{1}{2\pi} \int_0^{2\pi} X(\omega + 2\pi k/LKT)G^*(\omega)e^{-j\omega mMLT}d\omega$$

where the last line follows from Plancherel formula. Using the essentially band-limited assumption on $g(t)$, this is well approximated by:

$$s_k \approx \frac{1}{2\pi} \int_0^{2\pi/LT} X(\omega + 2\pi k/LKT)G^*(\omega)e^{-j\omega mMLT}d\omega \quad (5)$$

2.1.2. Sub-Nyquist TF Representations

Let us now consider the *discrete* TF representation for a single sub-Nyquist coset channel signal, $z_i[n]$. Since $g(t)$ is essentially band-limited, we can use the discrete time sampled atoms:

$$g_{m,k}[n] = g[n - mM]e^{j2\pi kn/K} = g((n - mM)LT)e^{j2\pi kn/K}. \quad (6)$$

The discrete time TF coefficients are given by:

$$r_{m,k}^{(i)} := \langle z_i[n], g_{m,k}[n] \rangle$$
$$= \sum_{n=mM}^{N+mM-1} z_i[n]g^*[n - mM]e^{-j2\pi kn/K}$$
$$= \frac{LT}{2\pi} \int_0^{\frac{2\pi}{LT}} Z_i(e^{j(\omega LT + 2\pi k/K)})G_d^*(e^{j\omega LT})e^{-j\omega mMLT}d\omega \quad (7)$$

where $G_d(e^{j\omega LT})$ is the DTFT of $g[n]$ and satisfies:

$$G_d(e^{j\omega LT}) = \frac{1}{LT} \sum_{k=-\infty}^{\infty} G(\omega - 2\pi k/LT)$$
$$\approx \frac{1}{LT}G(\omega), \quad 0 \leq \omega < 2\pi/LT \qquad (8)$$

where the approximation follows from the essentially band-limited assumption. From (2) we can also write:

$$Z_i(e^{j\omega LT + 2\pi jk/K}) =$$
$$\frac{1}{LT} \sum_{l=-\lceil \frac{k}{K} \rceil}^{\lceil L-1-\frac{k}{K} \rceil} e^{j2\pi l c_i/L} X(\omega + 2\pi k/LKT - 2\pi l/LT) \quad (9)$$

by shifting the window in frequency over which we evaluate Z_i. Substituting (9) into (7) we get:

$$r_{m,k}^{(i)} \approx \frac{1}{2\pi LT} \sum_{l=-\lceil \frac{k}{K} \rceil}^{\lceil L-1-\frac{k}{K} \rceil} \int_0^{\frac{2\pi}{LT}} e^{j2\pi l c_i/L}.$$
$$X(\omega + 2\pi k/LKT - 2\pi l/LT)G^*(\omega)e^{-j\omega mMLT}d\omega$$
$$\approx \frac{1}{LT} \sum_l e^{j2\pi l c_i/L} s_{m,k+lK} \qquad (10)$$

where the final approximation is good as long as the out-of-band aliasing effects are negligible. Discrete TF coefficients are therefore well approximated as the *sum of full band TF coefficients weighted by a coset dependent phase term*.

2.2. Sub-Nyquist Reconstruction Algorithm

In order to proceed further, we introduce the following definition.

Definition 1 (Approximate Disjoint Aliased Support (ADAS)). *We say that $x(t)$ has* approximate disjoint aliased support *in a given TF representation if each sub-Nyquist discrete TF coefficient is dominated only by a single full band TF coefficient such that:*

$$r_{m,k}^{(i)} \approx \frac{1}{LT} e^{j2\pi l_{m,k} c_i / L} s_{m,k+l_{m,k}K}, \qquad (11)$$

where $l = l_{m,k}$ is now a function of the sub-Nyquist TF position, (m,k), $m = 0, 1, \ldots$ and $k = 0, \ldots K - 1$.

We proceed by assuming that $x(t)$ has ADAS. This concept is similar to the notion of approximate disjoint orthogonality used in blind source separation [12][1].

While *a priori* we do not know the subband to which $r_{m,k}^{(i)}$ should be associated, we can solve this by comparing the discrete TF representations for each coset channel. However, first we need to detect the significant TF coefficients containing signal components. We can then determine the correct subbands $l_{m,k}$ with which to associate them. Both these tasks can be accomplished by considering the cross channel vector of TF coefficients, $\mathbf{r}_{m,k} = [r_{m,k}^{(1)}, \ldots, r_{m,k}^{(q)}]^T$.

2.2.1. Detection

Under the ADAS assumption one would expect that the TF for each channel would have a similar magnitude,

$$\|r_{m,k}^{(1)}\|_2 \approx \|r_{m,k}^{(2)}\|_2 \ldots \approx \|r_{m,k}^{(q)}\|_2.$$

It is also reasonable to assume that the noise effects on different channels are mutually independent since the sampling times are temporally distinct. One possible detection strategy is therefore to define the coefficient as significant as long as the magnitude $\|\mathbf{r}_{m,k}\|_2 > \tau$, where the threshold value τ can be chosen to achieve a constant false alarm rate.

2.2.2. Subband Classification

Once a coefficient vector has been detected as significant it needs to be assigned to a subband. This can be considered as a classification or decoding task. Under a Gaussian noise assumption, the optimal classifier is achieved by maximising the absolute inner product between $\mathbf{r}_{m,k}$ and the phase vector: $\theta(l) = [e^{j2\pi c_1 l / L}, \ldots, e^{j2\pi c_q l / L}]^T$ for the lth subband:

$$\hat{l}_{m,k} = \underset{l_k}{\operatorname{argmax}} |\sum_{i=1}^{q} r_{m,k}^{(i)} e^{-j2\pi c_i l_k / L}|^2$$

This classification is uniquely defined as long as the L phase vectors $\theta(l)$, $l = 0, \ldots, L - 1$ are all distinct, which can be achieved with as few as $q = 2$ coset channels!

[1]However, unlike in the DUET algorithm [12] where a delay is approximately with a frequency dependent phase shift, here the phase shift is constant across the aliased band and therefore a much broader class of TF representations can be used.

2.2.3. Reconstruction

Combining the detection and subband classification we can finally estimate the full band TF representation, $s_{m,k}$, as follows:

$$\hat{s}_{m,k+pK} = \begin{cases} \frac{LT}{q} \sum_{i=1}^{q} r_{m,k}^{(i)} e^{-j2\pi c_i \hat{l}_{m,k} / L}, & \text{if } p = \hat{l}_{m,k} \\ 0 & \text{otherwise} \end{cases}$$

for $m \geq 0$, $k = 0, \ldots, K - 1$ and $p = 0, \ldots, L - 1$. Full band time domain reconstruction, if desired, can then be achieved by applying an inverse full band TF transform to the coefficients, $\hat{s}_{m,k}$.

2.3. Optimal MC Sampling Delay Selection

The MC delays can be chosen to minimise the probability of incorrect subband classification. Under the ADAS assumption, optimal subband classification is achieved by sampling delays that are associated with harmonic frames with minimal coherence, μ, defined as [13]:

$$\mu = \max_{l \neq l'} |\langle \theta(l), \theta(l') \rangle|^2.$$

A benefit of using more MC channels is that the coherence, μ, can be reduced towards the optimal Welch bound [14] associated with Equiangular Tight Frames (ETF) [15]. There does not exist an optimal frame for an arbitrary down-sampling factor and a number of multi-coset channels. However, it has been shown that there exist Harmonic ETF, (HETF), when $q = p + 1$ and $L = q^2 - q + 1$, where p is a prime number [16]. For small numbers of channels, the delay sequence which generates a HETF, can be determined through an exhaustive search. For the simplest case of $q = 2$, a HETF is not generally attainable but any integer delay for c_2 (fixing $c_1 = 0$) achieves the minimal coherence as long as the greatest common divider of c_2 and L is $c_2 L$. For example this is trivially met by choosing $c_2 = 1$.
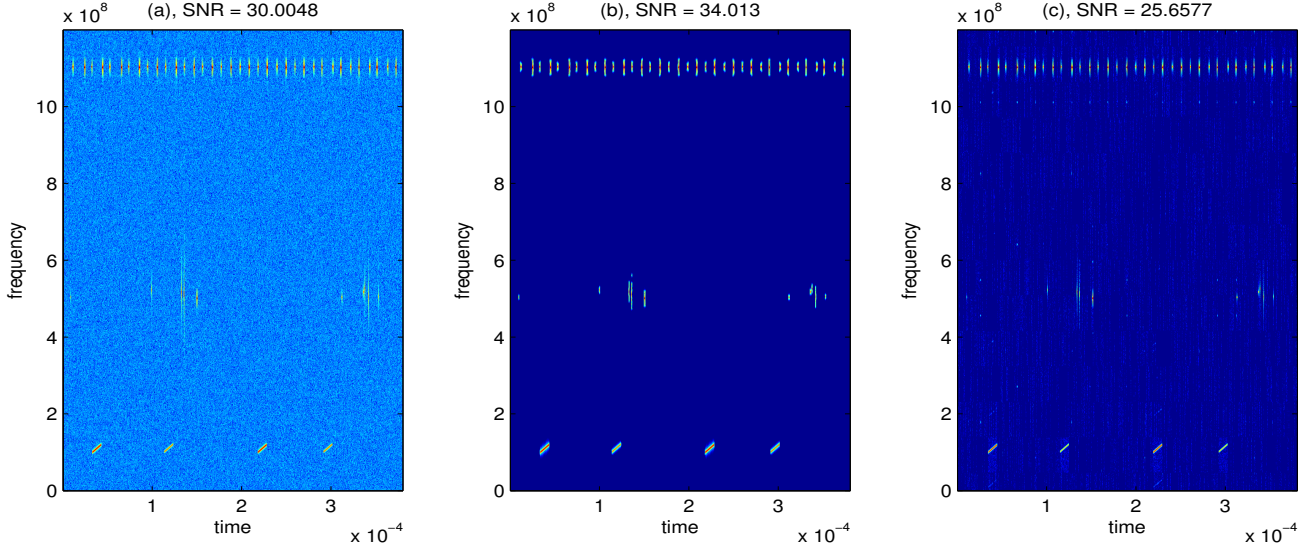
For the implementation of a DFD, we need to filter the signals with a truncated shifted-sinc function, which introduces some distortion, because of the non-ideal filtering. As we use linear TF transforms and DFD filters, which are associative, we can combine them to minimise the distortion. As the kernel of TF transform is available in the continuous domain, TF transformation of the DFD filter, generates a new set of shifted TF atoms, *i.e.*,

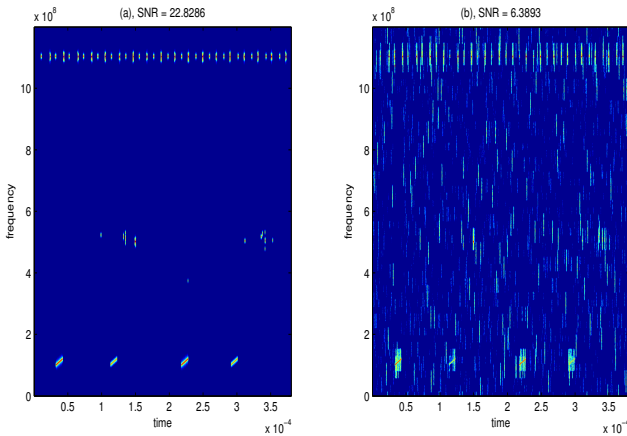$$g_{m,k}^{c_i}[n] = g(((n - mM)L + c_i)T) e^{\frac{j2\pi k}{KL}(nL + c_i)}.$$

The new transform can be *approximately* implemented by shifting the phase of $g_{m,k}[n]$, with $c_i T$. This technique has been used in the simulations of this paper.

3. SIMULATIONS

To evaluate the performance of LoCoMC, we set up a synthetic and a simulated Radar ESM experiments. We first used a chirped signal with starting and ending frequencies $[100MHz, 1.2GHz]$, in a period of $0.5 ms$. The magnitude of chirp signal was selected 17.78, while it was windowed with a Tukey window. A normal noise with $\sigma = 0.5$ was then added to the signal, *i.e.* SNR of the noisy signal was almost $30dB$. A down-sampling factor of $L = 13$, $q = 4$ multi-coset channels and $\tau = \sigma$ inv-$\chi^2(1 - 10^{-5}, q)$ were used in the proposed method, where inv-$\chi^2(\cdot, q)$ is the inverse chi-square *cdf*, with q degrees of freedom. We used a Hanning window [17] of size $2L\lfloor 512/L \rfloor$ in the STFT transform, where $\lfloor \cdot \rceil$ represents the closest integer. Delays of the system were selected $c = [0, 1, 4, 6]$, which generate a HETF. The reconstructed signal has a $34dB$ SNR,

Fig. 2. ESM signal reconstruction using LoCoMC (b) and windowed MUSIC (c) ($L = 13, q = 4$). The original noisy signal is shown in (a).

Fig. 3. ESM signal reconstruction using LoCoMC (a) and windowed MUSIC (b) ($L = 52, q = 4$).

which is higher than the noisy signal, due to the algorithm denoising behaviour.

We now compare the performance of LoCoMC with a canonical MC sampling algorithm, using MUSIC [2]. See also [18] for a short tutorial on the MC sampling technique. This approach assumes only a few frequency channels is active. In order to apply this to ESM signals with a dynamically changing spectrum, we apply a sliding window, of the size earlier used in STFT, to the signal. While the structure of input signal has been preserved using such a windowed MUSIC reconstruction technique, the SNR of the recovered signal, *i.e.* $25.66dB$, is much less than LoCoMC method.

In the second experiment, we used a set of RF pulses taken from a Radar ESM simulation[2]. A very similar setting to the previous experiment, was used in the sampling process, where we considered a $1.2GHz$ band, centred at 10 GHz, and demodulated to the baseband. The noisy ESM signal (left), reconstructed using LoCoMC (middle) and the reconstructed using a windowed MUSIC algorithm

(right) are shown in Figure 2. We can see the structure of the input signal, which is completely preserved in the reconstruction. However, much more false alarms appeared, when a windowed MUSIC was used, while the SNR was also almost $7dB$ lower than LoCoMC.

Next we evaluate the sampling performance for a larger undersampling ratio. We therefore used an undersampling of $L = 52$, which gives an average undersampling of 13. Using the previously selected set of delays, we no longer have a Harmonic ETF, in the subband classification step. We show the reconstructed signals with LoCoMC and windowed MUSIC techniques, respectively, in the left and right panels of Figure 3. Although, most of the pulses are recovered using LoCoMC from only 4 channels of highly aliased signals, some small pulses are missing, due to the "noise folding effect" discussed in [19]. We thus see that the signal is noise limited rather than sparsity limited in our system. On the other hand, the reconstructed signal by windowed MUSIC has a lower SNR, some misaligned pulses and many more false alarms.

4. CONCLUSION

A low complexity sub-Nyquist sampling technique was introduced in this paper, which can sample TF-sparse signals. The algorithm is based on the multi-coset sampling digitiser and a different algorithm for the signal reconstruction. As the method assumes a different sparse signal model, *i.e.* ADAS, it can out perform the canonical MC reconstruction techniques, for such signals. We showed that the new signal model fits very well to the Radar ESM signals, and we therefore yielded some SNR improvement and less false alarms.

As the future work, it is necessary to characterise the model mismatch robustness of the algorithm. To this end, we need to consider the analog design tolerance and the signal model mismatch. Which TF transform best fits to the ESM signals, *e.g.* an overcomplete transform like Chirplet [20], is also left for the future.

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