Compressed Sensing for Analog-to-Information Processing

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Analog Compressive Sampling

Random Demodulator
[Tropp et al. 2009]

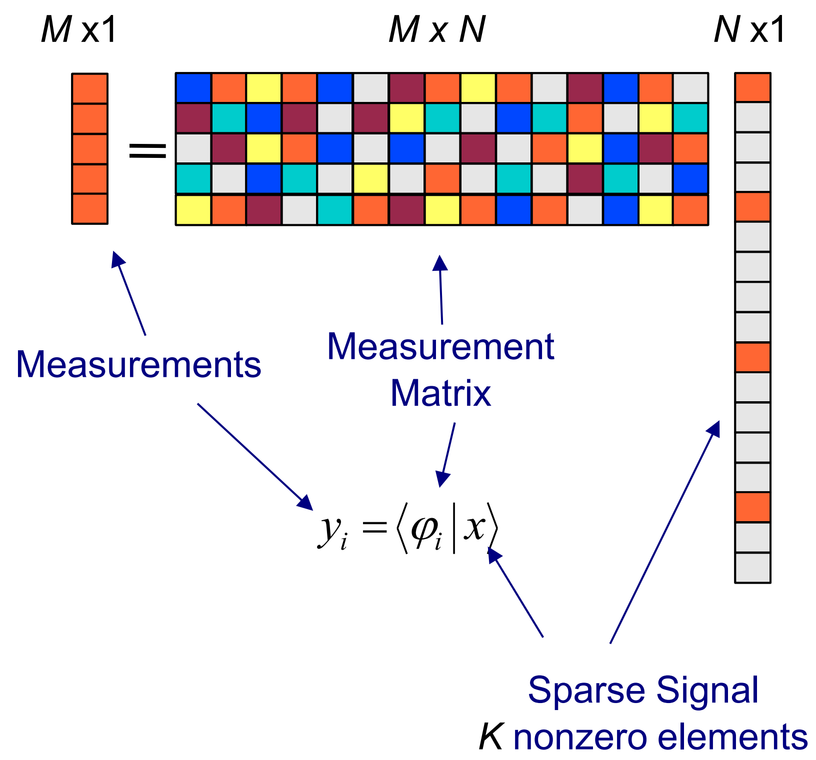
Multi-coset sampler
[Feng and Bressler 1996]

Modulated Wideband Converter
[Mishali and Eldar 2009]

Signal reconstruction
Signal detection

Fundamentals of Compressed Sensing (CS)

- CS was originally introduced for the acquisition of the **digital** signals.
- The sparsity of the signal helps to recover it from small number of measurements.
- We saw efficient techniques for the signal recovery.
- What about Analogue CS?
- Which signal models can be used in an analog setting?

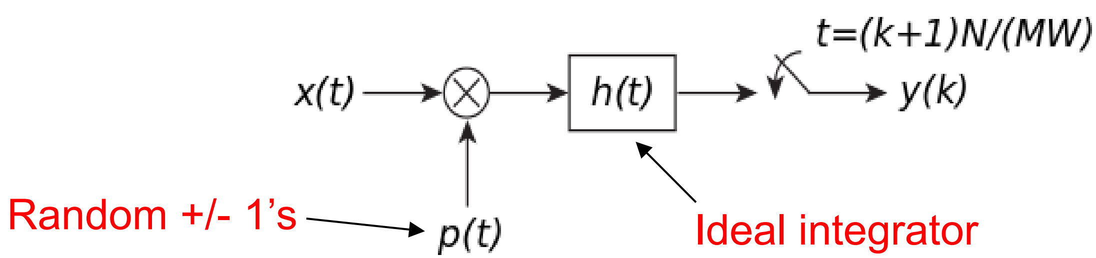


Random Demodulator

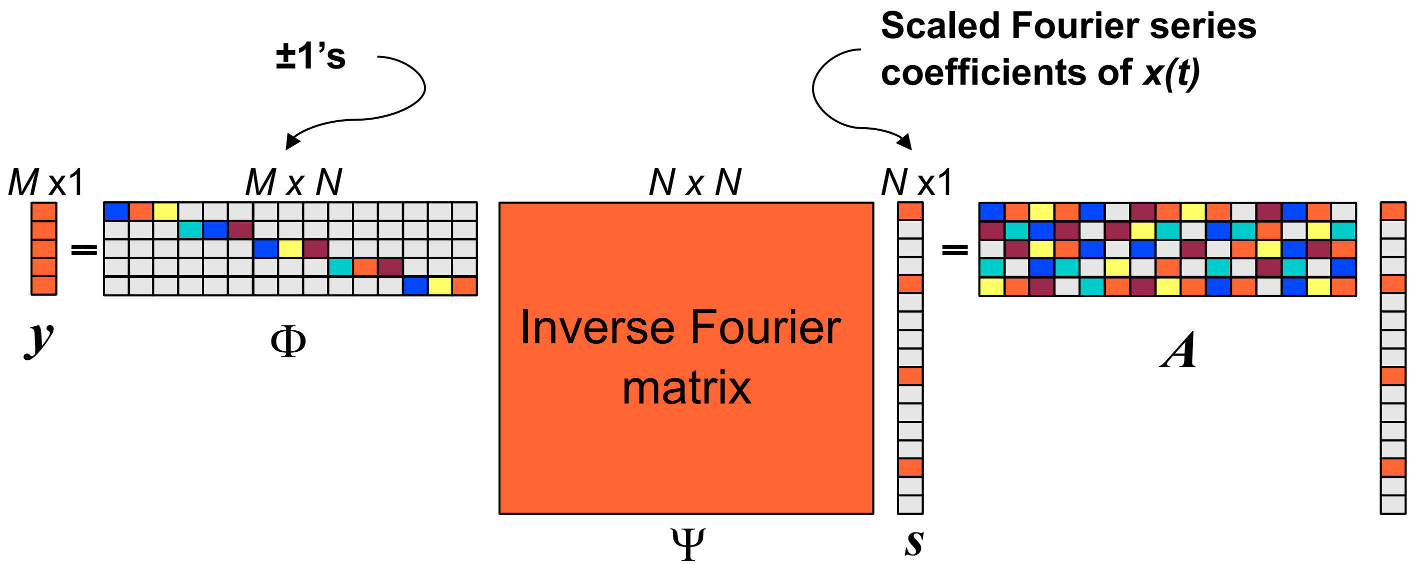
- Sub-Nyquist strategy for **sparse multitone signals** [Tropp et al. 2009]
  - **Bandlimited** *W/2* Hz
  - Periodic in time-domain $T_x = N/W$
  - Discrete, finite Fourier representation

- *p(t)* periodic extension of random signal taking values ±1
  - "Chipping" rate equal to Nyquist rate (*W* Hz)
  - Period equals period of *x(t)*

- Sample uniformly every $T_x/M = N/(MW)$ sec



- **Linear relationship** between time domain samples *y(k)* and Fourier series coefficients of *x(t)*
  - CS interpretation: Measurement matrix Φ and sparsifying matrix Ψ
  - Want to solve for Fourier coefficients of *x(t)* from samples *y(k)*
  - But we have an **underdetermined linear system**
  - Because of *x(t)*'s sparsity, **nonlinear optimization (CS)** techniques can recover the original spectrum of *x(t)* (under certain conditions for M,N, and K)



- Noiseless case
  - Optimisation problem $\hat{\boldsymbol{s}} = \arg\min \|\boldsymbol{v}\|_0 \text{ subject to } A\boldsymbol{v} = \boldsymbol{y}$

- Several existing methods solve these optimisation problems
  - Orthogonal Matching Pursuit (OMP) [Tropp 2006]
  - ℓ - minimisation (for the convex relaxed formulation)
  - Iterative Hard Thresholding [Blumensath and Davies 2008]

- For ℓ_1- minimisation, one can recover $\boldsymbol{s}$ with high probability if

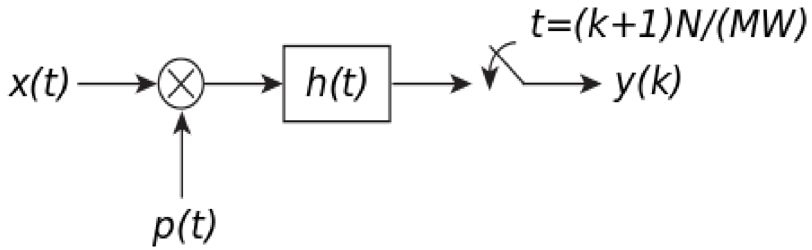
$$M \geq c\left(K \log WT_x + (\log WT_x)^3\right)$$

- Noisy case
  - Optimisation problem $\hat{\boldsymbol{s}} = \arg\min \|\boldsymbol{v}\|_0 \text{ subject to } \|A\boldsymbol{v} - \boldsymbol{y}\| \leq \varepsilon$
  - To obtain an estimate comparable to best K approximant

$$M \geq cK \log^6 WT_x$$

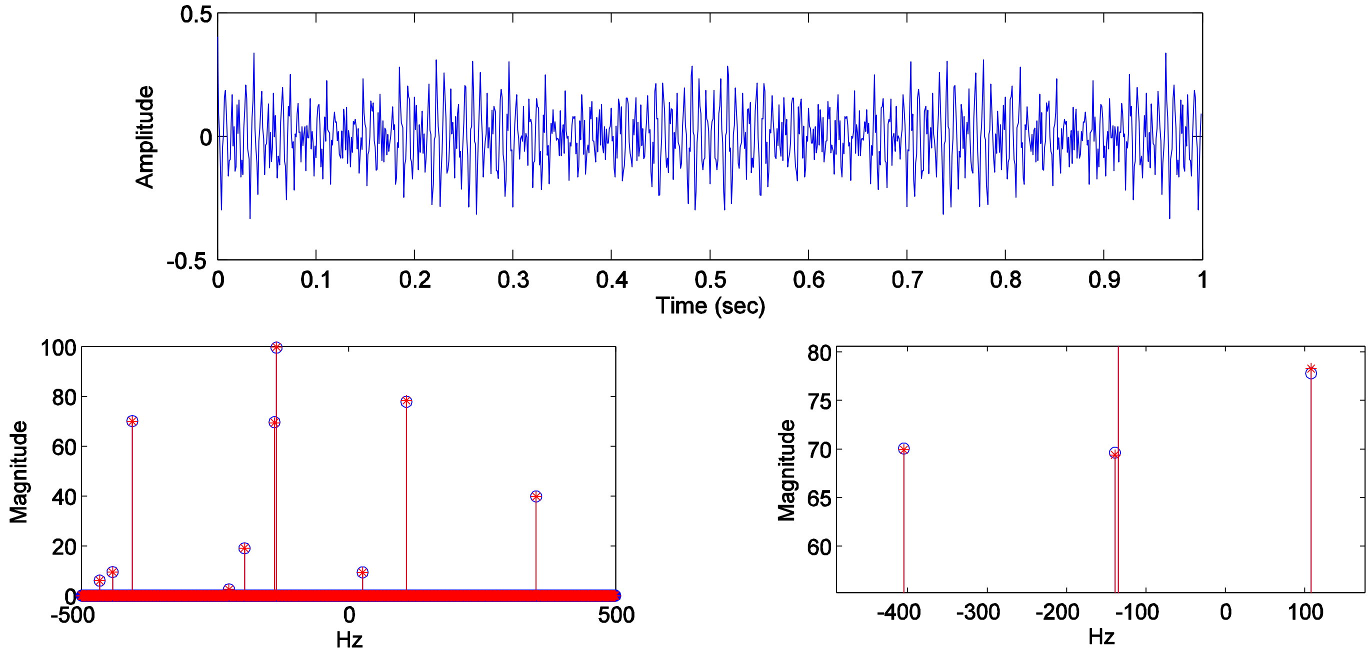
- RD components from CS perspective
  - *p(t)* and integrator provide measurement (sensing) mechanism
  - Sparsifying (Fourier) matrix results from signal model
  - Sampling simply provides discrete time-domain output

- No useful frequency domain interpretation
  - Multiplication by *p(t)* and sub-Nyquist sampling alias (mix) the signal but frequency domain analysis does not provide the insight it traditional does
  - Unlike Multi-coset sampler



Random Demodulator Example

- Sampled sparse multitone signal (10 tones) bandlimited to 500 Hz
- Random amplitudes and spectral locations
- Sampling rate: 1/10 of Nyquist (100 Hz)
- Recovered signal using orthogonal matching pursuit (OMP)

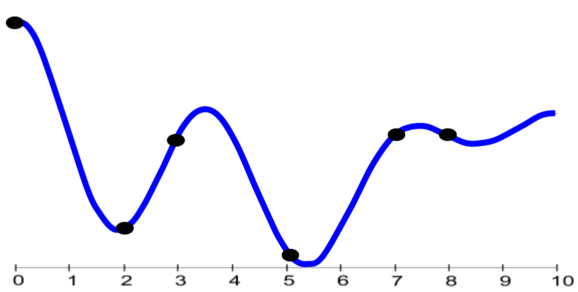


Multi-Coset Sampling (MC)

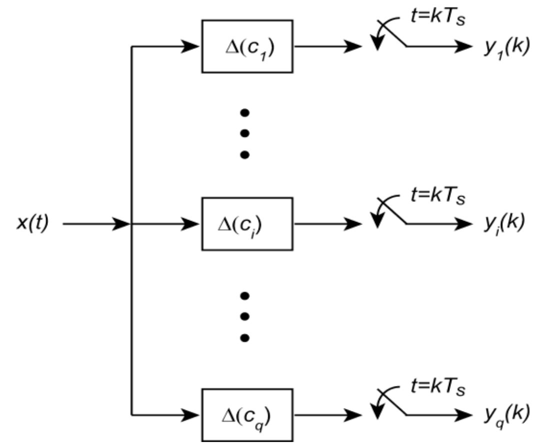
- Non-uniform sampling technique for **sparse multiband signals** [Feng and Bressler 1996]

- **Sparse multiband signals**
  - Bandlimited to W/2 Hz
  - ***K* occupied bands** of maximum bandwidth *B* Hz
  - Sparse – spectral occupancy small $KB << W$

- Sample *x(t)* at time instances $t = (kL + c_i)\frac{1}{W}$
  - Set of time delays $\{c_i\}_{i=1}^{q}$ is called the **sampling pattern**
  - Collect *q* samples from every *L* Nyquist periods

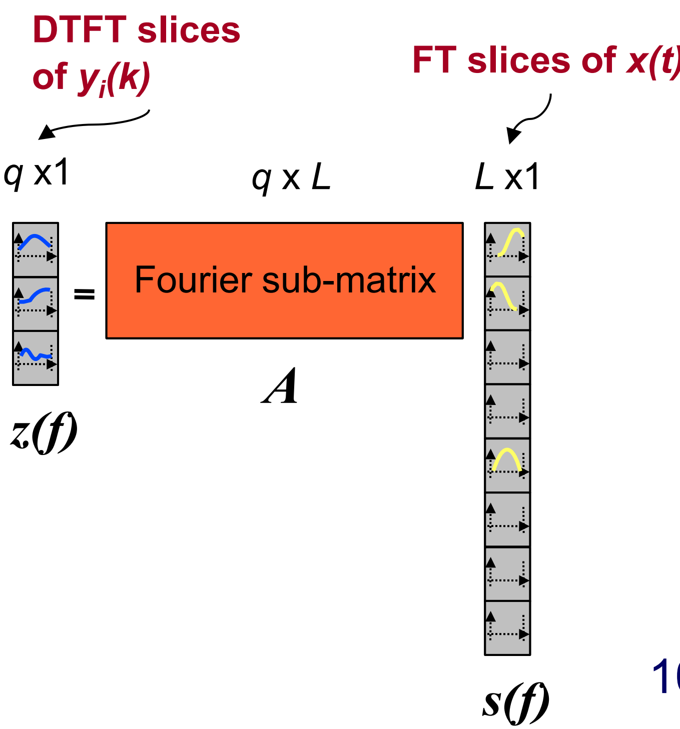


- Conceptually, MC sampling can be thought of as a multichannel system
  - Branch *i* delays *x(t)* by c_i
  - Each channel samples at the same rate *W*/*L* Hz



- Linear relationship between the DTFT of $y_i(k)$, $i=1,\ldots q$, and the spectral slices of *x(t)*
  - L controls spectral resolution (width of slices in *s(f)*)
  - Underdetermined linear system

- System involves linear combination of **functions** not scalar values



- To reconstruct, we need to identify unoccupied spectral slices (or equivalently spectral support of $x(t)$)
  - Reduces problem dimension so that linear system can be inverted

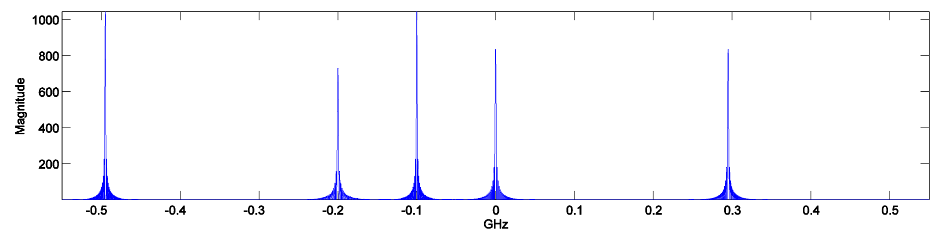
- Reduce dimension using a modified MUSIC algorithm
  - Relies on estimating covariance matrix R of output samples
  - Eigen-decomposition of R divides ambient space into two orthogonal subspaces (**noise** and **signal+noise**)
  - Columns of A that **lie in the null space of noise subspace** identify occupied slices ("MUSIC spectrum")

- Different delays create linear system that allows one to undo destructive aliasing caused by sub-Nyquist sampling
  - Conditions on when this is possible depends on properties of A and the sampling pattern

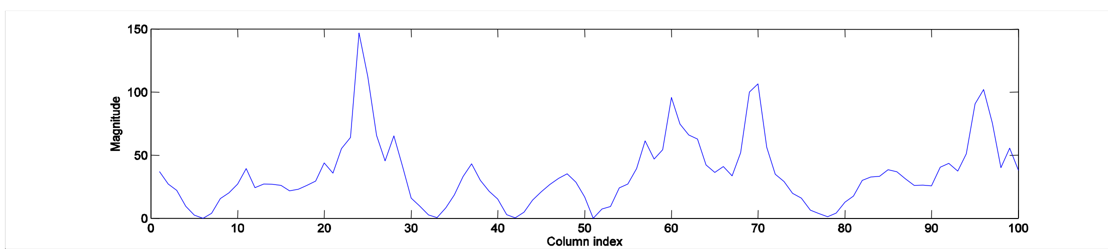
Multi-Coset Example

- Sampled simulated sparse multiband signal bandlimited to 1.1 GHz
- 5 bands
- Sampling pattern randomly chosen (uniformly)
- Sampling rate: 440 MHz (2.5 times slower than Nyquist)

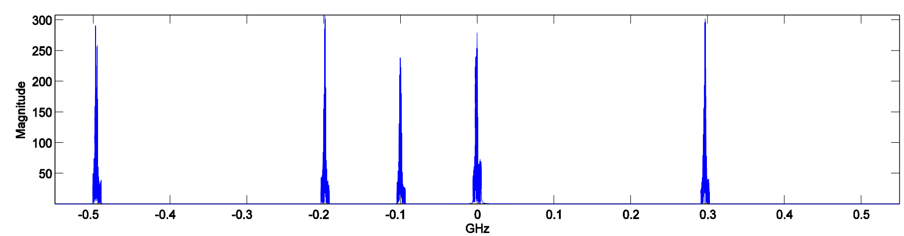
Original spectrum



"MUSIC spectrum"

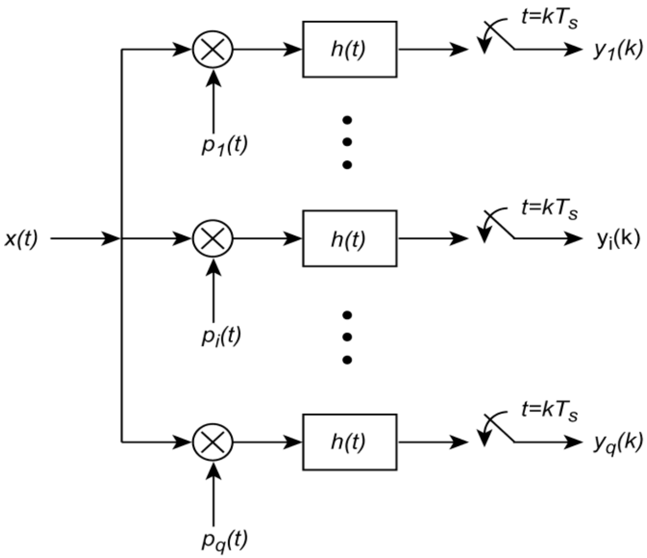


Reconstructed spectrum

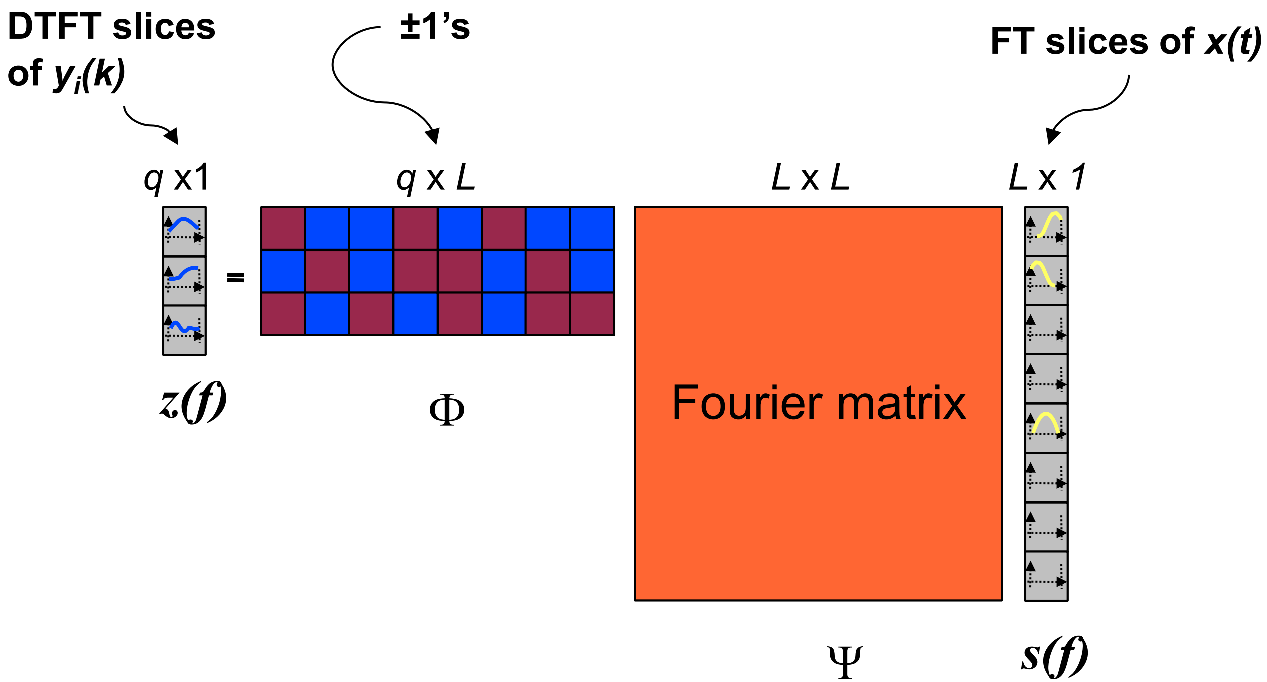


Modulated Wideband Converter (MWC)

- Sub-Nyquist sampling technique for sparse multiband signals [Mishali/Eldar 2009]
- Multichannel system
- Each channel multiplies *x(t)* by a periodic random square wave $p_i(t)$ taking values ±1
- The products $x(t)p_i(t)$ are filtered by ideal low pass filters each with cut-off frequency $1/2T_s$ (T_s is sampling frequency)
- Each channel samples uniformly at rate $1/T_s$
- Overall system sampling rate q/T_s



- Like MC sampler, **linear relationship** between DTFT of $y_i(k)$ and slices of the FT of $x(t)$
- Spectral resolution depends on the period of $p_i(t)$
- CS interpretation: Measurement matrix Φ and sparsifying matrix Ψ



- To reconstruct, identify spectral support, reduce dimension of linear system, and invert
  - Eigen-decomposition of covariance yields signal subspace
  - Solve associated linear system to find support

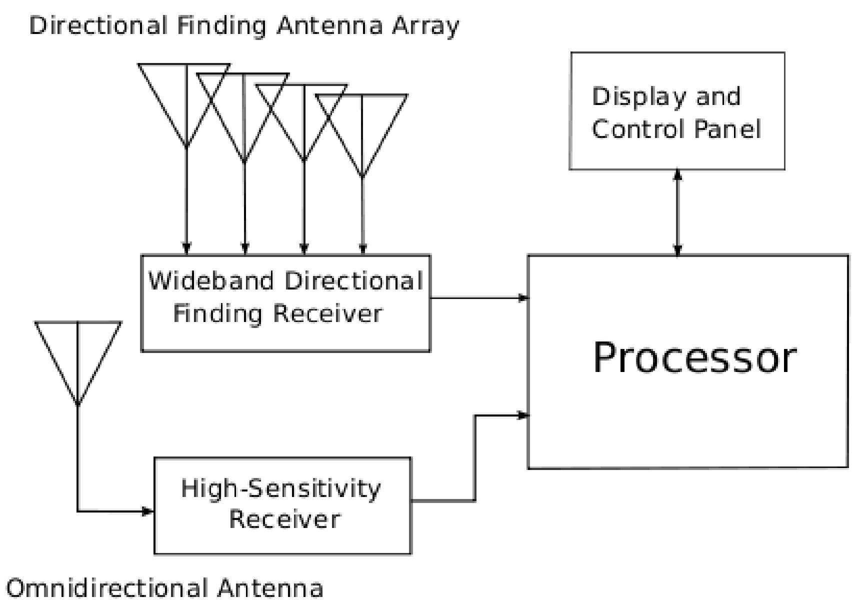
$$V(\lambda) = AU(\lambda)$$

Support of *U* same as support of *s(f)*

  - where $A = \Phi\Psi$ and *V* is column matrix of eigenvectors that span the signal subspace

- Conceptually, MWC relies on the same principles as MC
  - Both systems alias the spectrum of *x(t)* (MC uses sampling to alias, MWC uses multiplication by $p_i(t)$)
  - Both linearly relate the spectral slices of *x(t)* to the DTFT of $y_i(k)$
  - Both schemes invert the linear system by reducing the problem dimension (finding the support of *s(f)*)

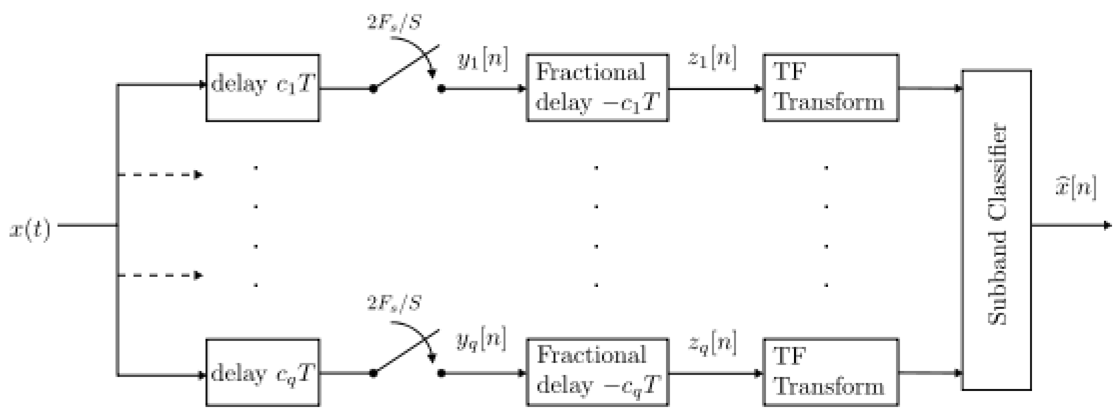
A Practical Application:
Sub-Nyquist Electronic Surveillance



- Needs a wide-bandwidth ADC for the front end.
- A single unit ADC to cover whole bandwidth is not practical.
- Rapid Swept Superheterodyne Receiver is based on time-sharing technique.
- The goal is to have a low SWAP alternative.

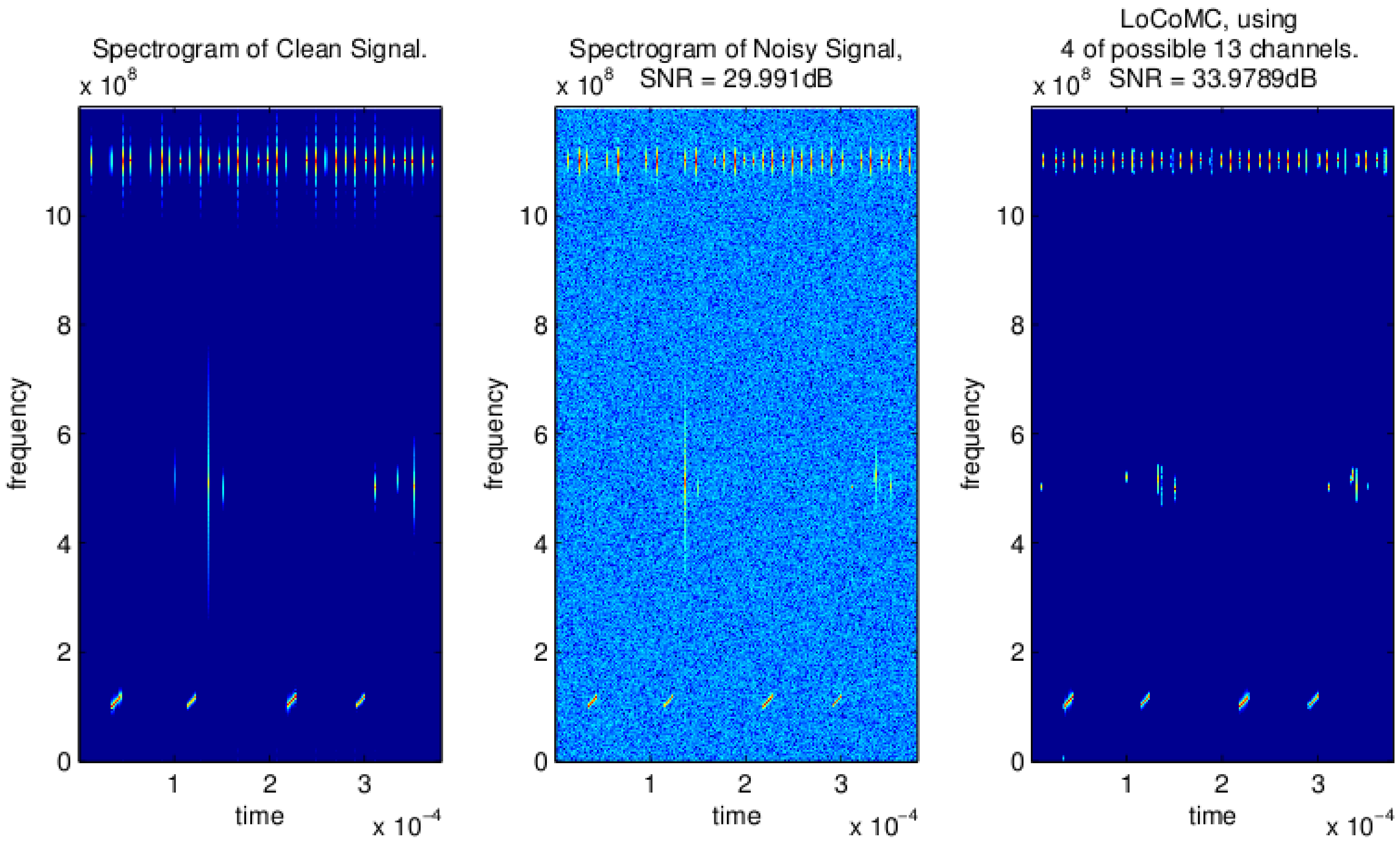
Low Complexity Multicoset Sampling

- MC has a simple analog front end.
- It uses a subspace method for signal reconstruction, *i.e*. high computation.
- A low-complexity algorithm can be presented using TF thresholding method.

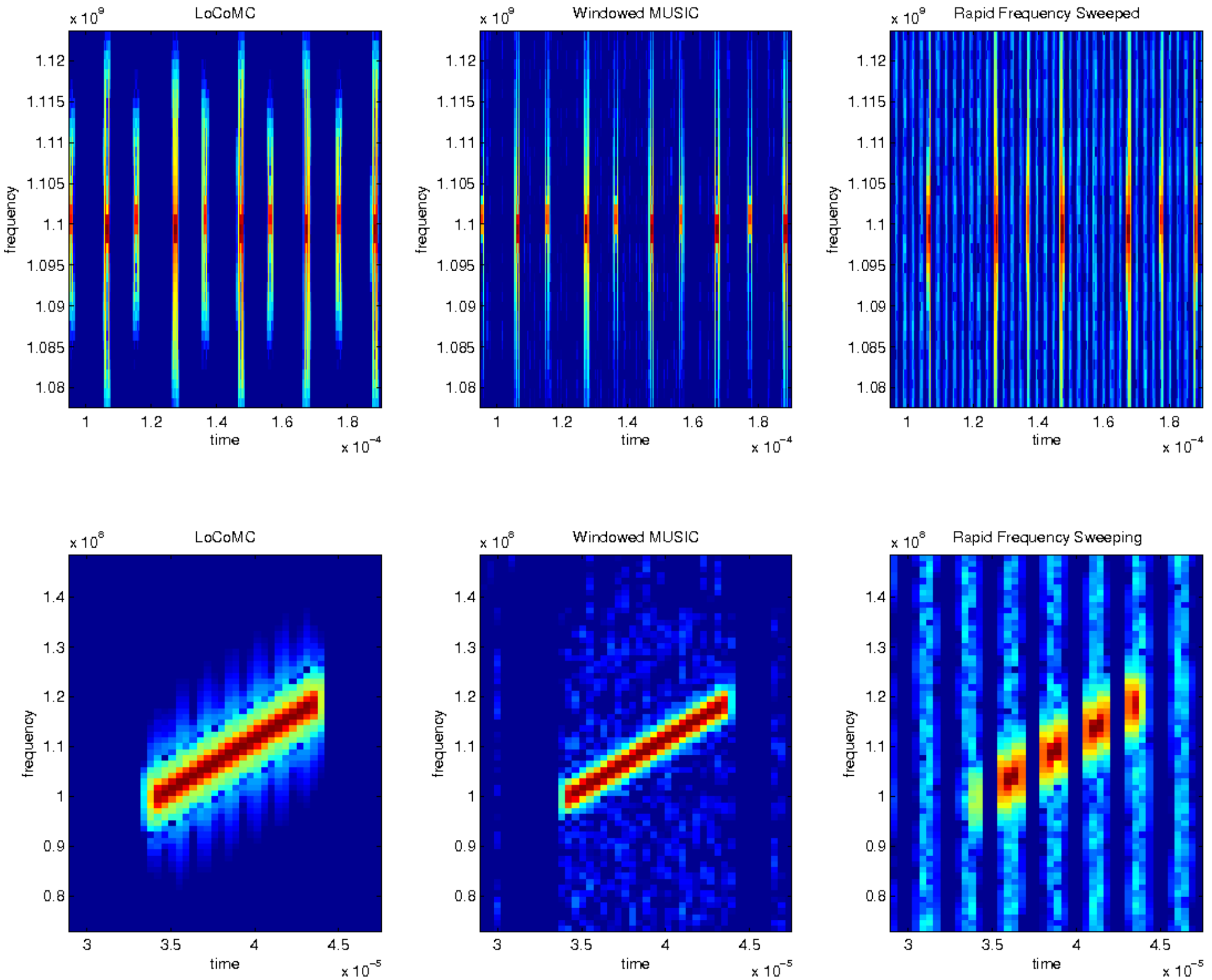


- While T/H is working in the sub-Nyquist rate, T part should be able to work in Nyquist rate.
- TF here is STFT and it is jointly implemented with the Fractional delay.
- Subband Classifier is a simple linear plus a comparator operator.

Radar Surveillance Signal Reconstruction



Comparison with MUSIC and RSSR



Summary

- Random demodulator
  - Directly inspired by CS
  - Sparse multitone signals
  - Formulation and signal reconstruction directly rely on standard CS algorithms

- Multi-coset sampler
  - Sub-Nyquist scheme that predates CS
  - Sparse multiband signals
  - Formulation and signal reconstruction have many close ties to CS

- Modulated Wideband Converter
  - Sub-Nyquist scheme based on the principles of MC sampling but leverages and incorporates CS and random demodulator ideas
  - Sparse multiband signals