Maximum Likelihood Sub-Arrayed MIMO Radar Receiver

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***Abstract*—There has been recent interest in MIMO radars which employ sub-arrays at the transmitter. This architecture allows a trade-off between the coherent beamforming gain of a phased array antenna and the transmit beampattern diversity offered by a MIMO radar with full transmit diversity, while maintaining a relatively simple implementation in the transmit hardware. In this work we study the maximum likelihood (ML) receiver for a transmit sub-arrayed MIMO radar, for estimating the direction of arrival of multiple targets. We develop the likelihood function for an orthogonal sub-arrayed MIMO radar for receiver architectures with and without a matched-filter bank. The maximisation of these likelihood functions form multi-dimensional optimisation problems which can be solved using a variety of sub-optimal numerical algorithms.**

***Index Terms*—MIMO Radar, Estimation, Maximum Likelihood**

I. Introduction

Orthogonal MIMO radars are able to create an extended aperture, which is formed due to the separable time delays of the transmitted signals impinging on the radar's receiver [1]. Sub-arrayed MIMO radars have recently received some interest from researchers [2], [3], [4], due to the trade-off which is offered between beamforming gain and transmit beampattern diversity. The work of [2] concentrates on the optimisation of the sub-array geometry and transmit sub-array steering vectors which is done in terms of the Cramér-Rao lower bound (CRB) on target direction of arrival (DOA) estimates for multiple point targets. We build on that work here, by studying the maximum likelihood (ML) receiver for an orthogonal MIMO radar which employs transmit sub-arrays. Maximum likelihood estimation is an important theoretical benchmark because it achieves the CRB, asymptotically with increasing signal to noise ratio (SNR) or number of snapshots. That is, an ML estimator can provide the lowest unbiased estimation error variance. Attaining the ML estimate for an estimation problem is often computationally intensive, so sub-optimum estimates need to be made for real-time processing.

The motivation for using sub-arrays for resolving multiple targets can be viewed from a monopulse perspective. It is well known that a single target can be accurately located utilising 2 squinted beams, though introducing a second target means that the two beams do not contain the information necessary to determine the target parameters. Multiple beams formed from transmit sub-arrays allow the target parameters to be determined, by effectively increasing the number of available equations to solve for a fixed number of unknowns. Practically, the monopulse approach to this problem is unfruitful as estimating high-order components of the beampatterns makes the process inaccurate and also a closed form solution to the set of equations formed is not available, so numerical procedures need to be employed. The maximum likelihood estimate effectively provides the solution to this set of equations.

A link may also be drawn to the amplitude modulation induced by targets illuminated by a scanning antenna described in [5], [6], [7]. In that system pulses from the radar transmitter are separated in the temporal processing, where as in the sub-arrayed MIMO radar the beams are distinguishable through the waveform coding. This allows data to be collected in a single snapshot rather than over several pulses which may be advantageous in environments with a short coherence time. An illustration of this concept is shown in figure 1.

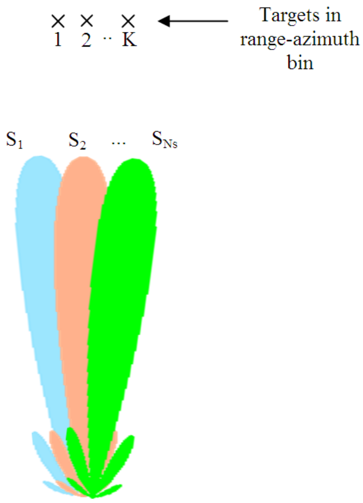

Fig. 1. Illustration of beampatterns formed from sub-arrays to locate multiple targets

The rest of the paper is organised as follows. In section II we describe the radar studied as part of this work, and develop the signal model used here. The likelihood function for this signal model is given in section III for the two receiver architectures with and without a matched-filter bank. In section IV we display simulation results for some simple scenarios, and draw our conclusions in section V.

II. Signal Model

We assume that the transmitter consists of a uniform linear array (ULA) of N_T elements with an antenna spacing of d_t, which is divided into N_S sub-arrays, each of which uses a different orthogonal narrowband signal from the set of waveforms $[\boldsymbol{s}_1 \cdots \boldsymbol{s}_{N_S}]$. The number of elements in each sub-array need not be equal, though throughout the work presented here it is assumed that this is the case. In general, however, the number of elements in the i^{th} sub-array may be denoted N_i. An example transmit array is displayed in figure 2. In this instance, the sub-arrays do not overlap as this is the architecture we will consider throughout the paper, though we note that the signal model and methods used here are equally applicable to the overlapping sub-array geometry, in for example [3].

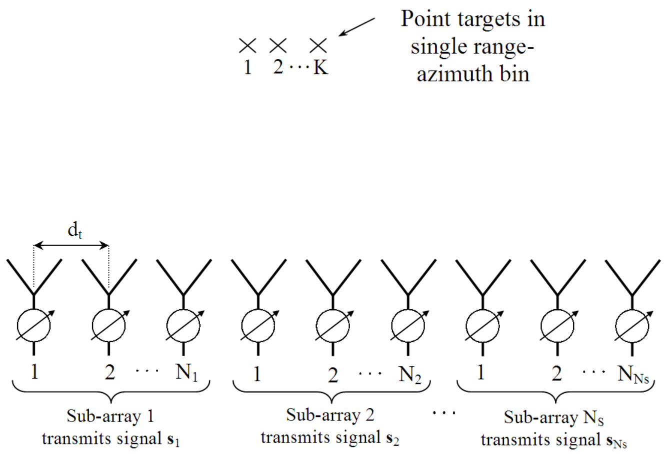

Fig. 2. Sub-arrayed MIMO Radar Transmit Architecture (sub-arrays may overlap)

The receiver consists of N_R receive elements also in a ULA with element spacing d_r. In order to separate out the orthogonal signals and therefore the phase delays from each of the targets, each antenna of the receive array may employ an associated matched-filter bank for the N_S transmitted signals, which gives an output signal vector of length $N_S N_R$.

A wavefront of a narrowband signal of wavelength λ originating from a target at an angle θ will have a phase delay at the n_r^{th} element of the receive array given by

$$u = e^{-j2\pi d_r n_r sin(\theta)/\lambda} \tag{1}$$

with $n_r = 0 \cdots N_R - 1$. The delay from the n_t^{th} element of the transmit array to the target, v, is defined similarly. It is considered that there are K point targets in a single range bin in the far-field of the radar array, then the response vector of the receive array to a plane wave from the k^{th} target can be written

$$\boldsymbol{u}(\theta_k) = [1 \; e^{-j2\pi d_r sin(\theta_k)/\lambda} \cdots e^{-j2\pi (N_R - 1) d_r sin(\theta_k)/\lambda}] \tag{2}$$

and again, the response vector for the transmit array is denoted as a vector $\boldsymbol{v}^T(\theta_k)$, and is defined similarly to $\boldsymbol{u}$ above. Throughout the paper, K will be assumed to be known, for example from previous track data. If K is unknown, the maximum likelihood receiver is still applicable, however an estimate of the number of targets would need to be included in the procedure, for example using a Bayesian information criterion metric as in [8].

The applied transmit beamforming vector is contained in a matrix $\boldsymbol{T} \in \mathbb{C}^{N_T \times N_S}$. Each column of $\boldsymbol{T}$ contains the steering vector for the relevant sub-array, padded with zeros to fit the extent of each sub-array. Intuitively, if the non-zero portions of $\boldsymbol{T}$ overlap, then so do the sub-arrays of the transmitter. With these definitions, the signal at the receive array can be written as

$$\boldsymbol{R} = \sum_{k=1}^{K} \boldsymbol{u}(\theta_k)\gamma_k \boldsymbol{v}^T(\theta_k)\boldsymbol{T}^* \boldsymbol{S} + \boldsymbol{E} \tag{3}$$

where γ_k is the complex amplitude of the k^{th} point target, * denotes the complex conjugate operation and $\boldsymbol{S} \in \mathbb{C}^{N_S \times L}$ is a matrix composed of samples of the baseband equivalent signals transmitted from the N_S transmit sub-arrays and L is the number of time samples. That is, $\boldsymbol{S} = [\boldsymbol{s}_1^T \cdots \boldsymbol{s}_{N_S}^T]^T$. We enforce a unitary power constraint on the signals in $\boldsymbol{S}$, that is $\mathbb{E}\{\boldsymbol{s}_i \boldsymbol{s}_i^H\} = 1 \;\; \forall i$, where $\mathbb{E}$ denotes the expectation. $\boldsymbol{E} \in \mathbb{C}^{N_R \times L}$ is a disturbance term which is assumed to be complex circularly symmetric white Gaussian noise with zero mean and covariance matrix $\boldsymbol{Q}$.

Equation (3) can be written in matrix form as

$$\boldsymbol{R} = \boldsymbol{U}(\boldsymbol{\theta})\boldsymbol{\Gamma}\boldsymbol{V}^T(\boldsymbol{\theta})\boldsymbol{T}^*\boldsymbol{S} + \boldsymbol{E} \tag{4}$$

with $\boldsymbol{U}(\boldsymbol{\theta}) = [\boldsymbol{u}(\theta_1) \; \cdots \; \boldsymbol{u}(\theta_K)]$, $\boldsymbol{V}(\boldsymbol{\theta}) = [\boldsymbol{v}(\theta_1) \; \cdots \; \boldsymbol{v}(\theta_K)]$, and $\boldsymbol{\Gamma}$ is a $K \times K$ diagonal matrix of the target complex amplitudes.

The transmit and receive array responses can be combined into a single matrix, the two-way antenna pattern, say $\boldsymbol{Y}(\boldsymbol{\theta})$, which depends on the target directions and the transmit sub-array steering vectors. Employing the matched filter bank at the receiver and remembering the unitary signal power constraint and with $\boldsymbol{\gamma}$ being a $K \times 1$ vector of the target complex amplitudes we may write down the matched filter bank output signal vector, $\boldsymbol{z} \in \mathbb{C}^{N_S N_R \times 1}$:

$$\boldsymbol{z} = \boldsymbol{Y}(\boldsymbol{\theta})\gamma + \text{vec}(\tilde{\boldsymbol{E}}) \tag{5}$$

where $\tilde{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{S}^H$ is the noise component of the signal matched filtered for each of the N_S signals and vec$(\cdot)$ is the vectorisation operation (columnwise stacking). $\tilde{\boldsymbol{E}}$ has covariance matrix $\tilde{\boldsymbol{Q}} = I_{N_S} \otimes \boldsymbol{Q}$, where I_{N_S} is the identity matrix of dimension N_S and $\otimes$ denotes the Kronecker product. This result is shown in appendix A. The dependence of $\boldsymbol{Y}$, $\boldsymbol{U}$ and $\boldsymbol{V}$ on $\boldsymbol{\theta}$ will be dropped from here onwards to ease notation.

III. The likelihood function

A. Matched-Filtering

To proceed, the signal model is translated to its likelihood function, which describes the statistical distribution of the received (and matched-filtered) signal, $\boldsymbol{z}$, given the variables of the signal model, that is, the target positions and complex amplitudes. The noise term, $\tilde{\boldsymbol{E}}$ is a multivariate Gaussian distribution, so letting $N_z = N_S N_R$ the likelihood function is given by

$$\begin{aligned} p_{\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma}}(\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma}) &= \frac{1}{(2\pi)^{N_z/2}|\tilde{\boldsymbol{Q}}|^{1/2}} \\ &\times \exp\left(-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{Y}\boldsymbol{\gamma})^H\tilde{\boldsymbol{Q}}^{-1}(\boldsymbol{z}-\boldsymbol{Y}\boldsymbol{\gamma})\right) \end{aligned}$$

The maximum likelihood estimate of the target parameters is then found by the maximisation of $p_{\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma}}(\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma})$ over the desired unknowns, or more formally:

$$\arg\max_{\boldsymbol{\theta},\boldsymbol{\gamma}} p_{\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma}}(\boldsymbol{z}|\boldsymbol{\theta};\boldsymbol{\gamma})$$

This is obviously equivalent to maximising the exponential term alone, and therefore the target parameter estimates are found from

$$\arg\min_{\boldsymbol{\theta},\boldsymbol{\gamma}} (\boldsymbol{z}-\boldsymbol{Y}\boldsymbol{\gamma})^H\tilde{\boldsymbol{Q}}^{-1}(\boldsymbol{z}-\boldsymbol{Y}\boldsymbol{\gamma}) \quad (6)$$

The maximum likelihood estimate for a model such as this is developed in [9] and is applied to this estimation problem here in appendix B. The maximum likelihood estimates of the angles and complex amplitudes of the targets are given by:

$$\hat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}} \boldsymbol{z}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}(\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} \quad (7)$$

$$\hat{\boldsymbol{\gamma}}_{ML} = (\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} \quad (8)$$

The desired quantities can be found by solving (7) which requires the optimisation over K-dimensions. The result is substituted into (8) to give $\hat{\boldsymbol{\gamma}}_{ML}$. Finding the maximum of equation (7) requires a K-dimensional search, which is arduous even for small values of K. To obtain the maximum, we use here a coarse grid search followed by an alternating projection method. Other sub-optimal methods for obtaining the maximum of the likelihood function, for example the incremental multi-parameter algorithm [10] or SAGE [11], would be equally applicable to this problem.

B. Without Matched-Filtering

While the matched-filter bank allows us to think in terms of the transmit beamspace, this process is not necessary for the maximum likelihood equations to be derived. Without the matched filters, the likelihood equations operate on the received data samples directly. The likelihood function is now given by

$$\begin{aligned} p_{\boldsymbol{R}|\boldsymbol{\theta};\boldsymbol{\gamma}}(\boldsymbol{R}|\boldsymbol{\theta};\boldsymbol{\gamma}) &= \prod_{l=0}^{L-1} \frac{1}{(2\pi)^{N_R/2}|\boldsymbol{Q}|^{1/2}} \\ &\times \exp(-\frac{1}{2}(\boldsymbol{R}_l - \boldsymbol{U}\Gamma\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}_l)^H \\ &\times \boldsymbol{Q}^{-1}(\boldsymbol{R}_l - \boldsymbol{U}\Gamma\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}_l)) \end{aligned} \quad (9)$$

where $\boldsymbol{R}_l$ and $\boldsymbol{S}_l$ denote the l^{th} receive array and signal sample vectors. The maximum likelihood equations in this receiver architecture are considered in appendix C.

The maximum likelihood estimate of $\boldsymbol{\theta}$ for the non-matched-filtered data is given by:

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{ML} &= \arg\max_{\boldsymbol{\theta}} \text{trace}[2\text{Re}\{\boldsymbol{R}^H\boldsymbol{Q}^{-1}\boldsymbol{U}\Gamma\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}\} \\ &+ \boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^*\Gamma^H\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{U}\Gamma\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}] \end{aligned} \quad (10)$$

with $\text{Re}\{\cdot\}$ denoting the real part of the argument and

$$\hat{\Gamma}_{ML} = (\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{R}\boldsymbol{S}^H\boldsymbol{T}^*\boldsymbol{V}^T(\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}\boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^*)^{-1} \quad (11)$$

When the matched-filter bank is not employed, there are L samples from the environment, though each of these samples has an SNR of $1/L$ that of the matched-filtered data sample. The receiver with the matched-filter bank will only have one range sample on which to make its estimate, but a much improved SNR over the receiver which does not matched-filter the received data. We would expect that having available a large number of samples would mitigate the reduction in SNR for the non-matched-filtered data. Another important aspect of employing the matched-filter bank is that a major benefit of matched-filtering is that it reduces the relative strength out of range bin contributions by other targets and clutter to the data. While this is not an issue for the idealised simulations shown here, practically this could significantly reduce the performance of the second receiver architecture.

IV. Simulation Results

In order to investigate the performance of the algorithms for attaining the maximum likelihood estimate, 1000 Monte Carlo test signals following the model in (4) were generated for a variety of target parameters. The physical radar parameters used are: $N_T = N_R = 24$ and $d_t = 1$, $d_r = N_S$. Figures 3 and 4 show the CRB and MLE for target 1, where two targets are separated by the angle displayed on the x-axis using 2 and 3 transmit sub-arrays. Increasing the number of sub-arrays allows a longer effective aperture to be produced and the estimation performance is improved. As the targets move closer than 0.5^o the bias in the target DOA estimates increases as the MLE procedure fails. Figure 3 shows that the ML procedure does not attain the CRB and this is because only one data sample is used in the estimation, and this could be overcome by utilising more transmit pulses. Figures 5 and 6 show the performance of the ML procedure for the two receiver architectures with and without the matched-filter bank for $L = 32$ and $L = 64$. With 64 data samples, the receiver architecture which does not use a matched-filter bank meets the CRB. Reducing the number of available samples to 32, however, is not enough for the MLE to overcome the low SNR and the DOA estimation performance is severely reduced.

V. Conclusions

The maximum likelihood receiver for an orthogonal sub-arrayed MIMO radar has been investigated in this work and examples have been presented for the resolution of two targets. The likelihood equations which must be maximised have been derived for two receiver architectures, with and without matched-filter banks at the receive elements. When using a

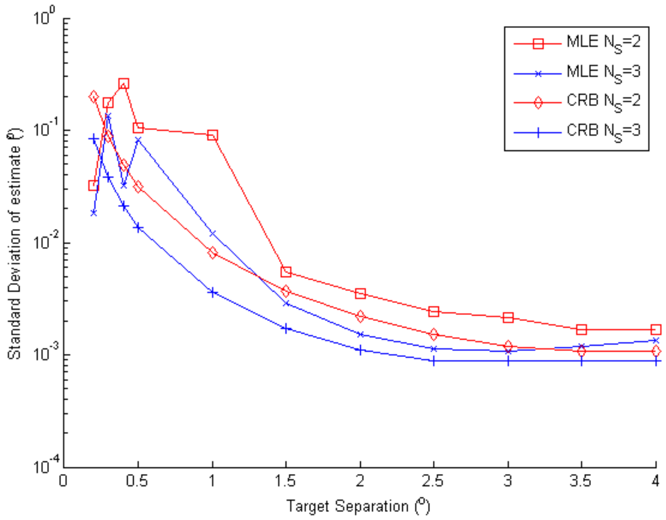

Fig. 3. Standard deviation and Root CRB of target 1 position estimate for $N_S = 2$ and $N_S = 3$

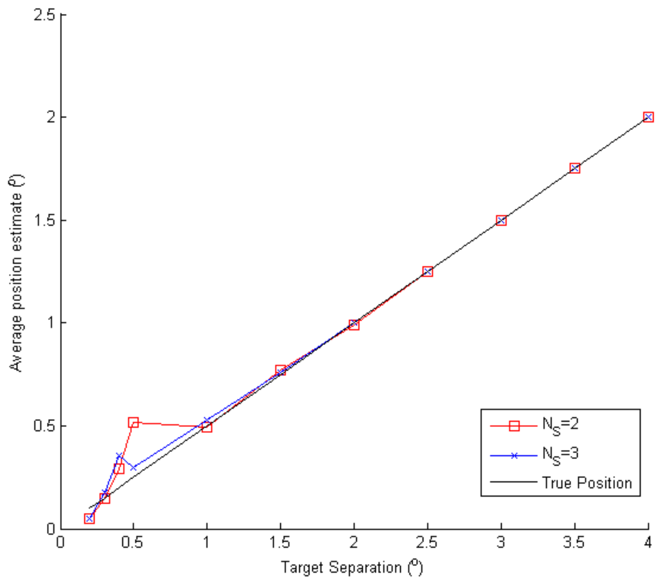

Fig. 4. Target 1 position estimate and true position for $N_S = 2$ and $N_S = 3$

matched-filter bank to separate out the signals such that the data is transformed into the transmit beamspace domain, there is a performance loss from the CRB caused by working only from 1 data sample, which could be remedied by employing more than one transmit pulse to gather further matched-filter samples, however, this removes the 'monopulse' operation of the radar. Without using a matched-filter bank there are more available data samples, though the penalty is lower signal to noise ratio. It has been seen that with enough data samples, the low SNR can be overcome, and the CRB was achieved by the sub-arrayed MIMO radar receiver. This neglects an important advantage of matched-filtering though, which reduces out of range-bin contributions from other targets

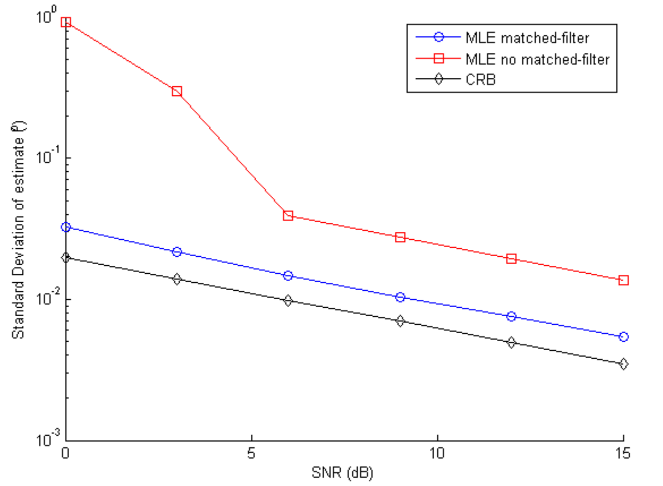

Fig. 5. Standard Deviation of MLE with and without matched filtering, $L = 32$

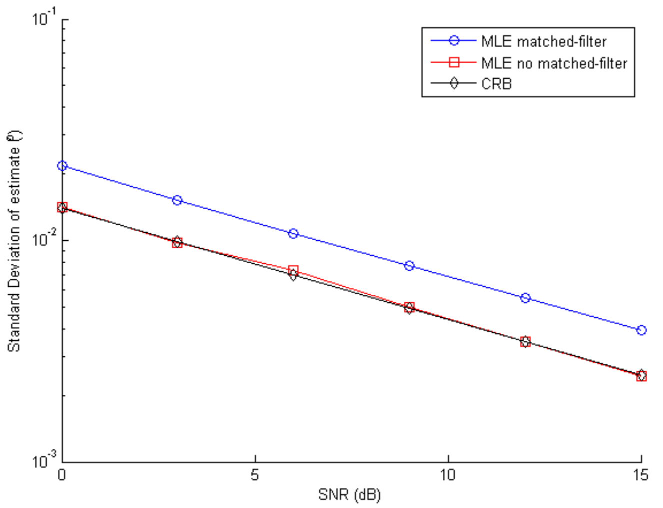

Fig. 6. Standard Deviation of MLE with and without matched filtering, $L = 64$

and clutter. Increasing the number of sub-arrays allows a longer effective aperture to be formed, so the two targets can be resolved more accurately.

Appendix A
Matched Filtered Noise Distribution

The matched filtered noise covariance matrix is

$$\tilde{\boldsymbol{Q}} = \mathbb{E}\{\text{vec}(\tilde{\boldsymbol{E}})\text{vec}(\tilde{\boldsymbol{E}})^H\} \tag{12}$$

where $\mathbb{E}\{\cdot\}$ denotes the expectation operator, and $\text{vec}(\cdot)$ is the vectorisation, or column-wise stacking, of a matrix. This can be written as

$$\tilde{\boldsymbol{Q}} = \mathbb{E}\{(\boldsymbol{S}^* \otimes I_{N_T})\text{vec}(\boldsymbol{E})\text{vec}(\boldsymbol{E})^H(\boldsymbol{S}^* \otimes I_{N_T})^H\} \tag{13}$$

where we have used the facts that

$$\text{vec}(\boldsymbol{AB}) = (\boldsymbol{B}^T \otimes I)\text{vec}(A)$$

and

$$\boldsymbol{A}\boldsymbol{B}^H = \boldsymbol{B}^H\boldsymbol{A}^H$$

Only E is a random process, so the two outer terms are not dependent on the expectation operator and can be brought outside of it. The expectation is equivalent to $I_L \otimes \boldsymbol{Q}$, and a further identity:

$$(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = (\boldsymbol{A}\boldsymbol{C}\otimes\boldsymbol{B}\boldsymbol{D})$$

which holds when the dimensions are appropriate for the matrix multiplications (which is true in this case), can be used to give the result that

$$\begin{aligned}\tilde{\boldsymbol{Q}} &= (\boldsymbol{S}^* I_L \boldsymbol{S}^T)\otimes(I_{N_T}\boldsymbol{Q}I_{N_T}) \\ &= I_{N_S}\otimes\boldsymbol{Q} \qquad (14)\end{aligned}$$

besause $\boldsymbol{S}^*\boldsymbol{S}^T = I_{N_S}$, which is the desired result.

APPENDIX B
MAXIMUM LIKELIHOOD EQUATIONS - MATCHED-FILTERED DATA

The function to be minimised, say $J_1(\boldsymbol{\theta},\boldsymbol{\gamma}) = (\boldsymbol{z} - \boldsymbol{Y}\boldsymbol{\gamma})^H\tilde{\boldsymbol{Q}}^{-1}(\boldsymbol{z} - \boldsymbol{Y}\boldsymbol{\gamma})$, is differentiated with respect to $\boldsymbol{\gamma}$ to find the stationary points of the function. The result proceeds as follows.

$$\begin{aligned}J_1(\boldsymbol{\theta},\boldsymbol{\gamma}) &= (\boldsymbol{z}^H - \boldsymbol{\gamma}^H\boldsymbol{Y}^H)\tilde{\boldsymbol{Q}}^{-1}(\boldsymbol{z}-\boldsymbol{Y}\boldsymbol{\gamma}) \\ &= \boldsymbol{z}\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} - \boldsymbol{\gamma}^H\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} - \boldsymbol{z}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}\boldsymbol{\gamma} \\ &\quad + \boldsymbol{\gamma}^H\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}\boldsymbol{\gamma}\end{aligned}$$

Then

$$J_1'(\boldsymbol{\theta},\boldsymbol{\gamma}) = \frac{\partial J_1(\boldsymbol{\theta},\boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}} = 2\boldsymbol{Y}\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}\boldsymbol{\gamma} - 2\boldsymbol{z}\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y} \qquad (15)$$

The minimum of the function $J_1(\boldsymbol{\theta},\boldsymbol{\gamma})$ occurs where $J_1'(\boldsymbol{\theta},\boldsymbol{\gamma}) = \boldsymbol{0}$ where $\boldsymbol{0}$ is a vector of zeros of the appropriate size, which leads to

$$2\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}\boldsymbol{\gamma} - 2\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} = \boldsymbol{0} \qquad (16)$$

which, with some simple manipulation shows that the minimum of the likelihood function, that is the maximum likelihood estimate, occurs at a point where

$$\hat{\boldsymbol{\gamma}}_{ML} = (\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} \qquad (17)$$

This can be substituted into the original ML equation to obtain an equation independent of the RCS of the targets.

$$\begin{aligned}J_1(\boldsymbol{\theta}) &= (\boldsymbol{z} - \boldsymbol{Y}(\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z})^H\tilde{\boldsymbol{Q}}^{-1}\times \\ &\quad (\boldsymbol{z} - \boldsymbol{Y}(\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z})\end{aligned}$$

which after some matrix algebra reduces to

$$J_1(\boldsymbol{\theta}) = \boldsymbol{z}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} - \boldsymbol{z}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}(\boldsymbol{Y}\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}^H)^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} \quad (18)$$

The first term does not affect the shape of the likelihood function, so does not affect the position of the ML estimate, and so the ML estimate of the target angles as found by minimising the second term, which is the same as maximising the term without the negation. Thus

$$\hat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}} \boldsymbol{z}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y}(\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{Y})^{-1}\boldsymbol{Y}^H\tilde{\boldsymbol{Q}}^{-1}\boldsymbol{z} \qquad (19)$$

APPENDIX C
MAXIMUM LIKELIHOOD EQUATIONS - NON-MATCHED-FILTERED DATA

Taking the natural logarithm of (9) and dropping terms which do not affect the ML estimate we have:

$$J_2(\boldsymbol{\theta},\boldsymbol{\Gamma}) = -\sum_{l=0}^{L-1}(\boldsymbol{R}_l - \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}_l)^H\boldsymbol{Q}^{-1}(\boldsymbol{R}_l - \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}_l))$$

This may be rewritten in terms of the trace operator and expanded as

$$\begin{aligned}J_2(\boldsymbol{\theta},\boldsymbol{\Gamma}) &= -\text{trace}[\boldsymbol{R}^H\boldsymbol{Q}^{-1}\boldsymbol{R} - \boldsymbol{R}^H\boldsymbol{Q}^{-1}\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S} \\ &\quad - \boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^*\boldsymbol{\Gamma}^H\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{R} \qquad (20) \\ &\quad + \boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^*\boldsymbol{\Gamma}^H\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}] \qquad (21)\end{aligned}$$

A necessary condition for the maximum of the function is that $\frac{\partial J_2(\boldsymbol{\theta},\boldsymbol{\Gamma})}{\partial\boldsymbol{\Gamma}} = \boldsymbol{0}$. So

$$\begin{aligned}\boldsymbol{0} &= 2\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T\boldsymbol{T}^*\boldsymbol{S}\boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^* \\ &\quad - 2\boldsymbol{U}^H\boldsymbol{Q}^{-1}\boldsymbol{R}\boldsymbol{S}^H\boldsymbol{T}^T\boldsymbol{V}^* \qquad (22)\end{aligned}$$

which can be rearranged to form the expression for $\boldsymbol{\Gamma}_{ML}$ in 11.

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