Optimal and Adaptive Filtering

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Optimal filter design

Figure 1: Optimal filtering scenario.

- $y(n)$: Observation related to a signal of interest $x(n)$.
- $h(n)$: The impulse response of an LTI estimator.
Optimal filter design

\[ y(n) \quad \rightarrow \quad h(n) \quad \rightarrow \quad \hat{x}(n) \]

- **Observation sequence**
- **Linear time invariant system**
- **Estimation**

**Figure 1**: Optimal filtering scenario.

- \( y(n) \): Observation related to a signal of interest \( x(n) \).
- \( h(n) \): The impulse response of an LTI estimator.
- Find \( h(n) \) with the best error performance:

\[
e(n) = x(n) - \hat{x}(n) = x(n) - h(n) * y(n)
\]

- The error performance is measured by the mean squared error (MSE)

\[
\xi = E \left[ (e(n))^2 \right].
\]
**Optimal filter design**

![Optimal filtering scenario](image)

**Figure 2:** Optimal filtering scenario.

- The MSE is a function of $h(n)$, i.e.,

$$
h = [\cdots, h(-2), h(-1), h(0), h(1), h(2), \cdots]$$

$$
\xi(h) = E \left[ (e(n))^2 \right] = E \left[ (x(n) - h(n) \ast y(n))^2 \right].
$$
The MSE is a function of $h(n)$, i.e.,

$$h = [\cdots, h(-2), h(-1), h(0), h(1), h(2), \cdots]$$

$$\xi(h) = E \left[ (e(n))^2 \right] = E \left[ (x(n) - h(n) * y(n))^2 \right].$$

Thus, optimal filtering problem is

$$h_{opt} = \arg\min_h \xi(h).$$
Application examples

1) Prediction, interpolation and smoothing of signals

\[
\begin{align*}
x(n) & \xrightarrow{z^{-d}} x'(n) = x(n - d) \\
\omega(n) & \xrightarrow{+} y(n) \\
h(n) & \xrightarrow{} \hat{x}(n)
\end{align*}
\]
Application examples

1) Prediction, interpolation and smoothing of signals

- Linear predictive coding (LPC) in speech processing.

\[ x(n) \xrightarrow{z^{-d}} x'(n) = x(n-d) \xrightarrow{\omega(n)} y(n) \xrightarrow{h(n)} \hat{x}(n) \]

(a) \( d = 1 \)
(b) \( d = -1 \)
(c) \( d = -1/2 \)

- Linear predictive coding (LPC) in speech processing.
Application examples

2) System identification

\[ y(n) = t(n) \]

\[ g(n) \]

\[ x(n) \]

\[ h(n) \]

\[ \hat{x}(n) \]

**Figure 3:** System identification using a training sequence \( t(n) \) from an ergodic and stationary ensemble.

- Echo cancellation in full duplex data transmission.
3) **Inverse System identification**

\[ x(n) = t(n) + \omega(n) \Rightarrow y(n) = g(n) + h(n) \Rightarrow \hat{x}(n) \]

**Figure 4:** Inverse system identification using \( x(n) \) as a training sequence.

- Channel equalisation in digital communication systems.
Optimal solution: Normal equations

Consider the MSE $\xi(h) = E[(e(n))^2]$

The optimal filter satisfies $\nabla \xi(h) |_{h_{opt}} = 0$. Equivalently, for all $j = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\frac{\partial \xi}{\partial h(j)} = E \left[ 2e(n) \frac{\partial e(n)}{\partial h(j)} \right]$$

$$= E \left[ 2e(n) \frac{\partial (x(n) - \sum_{i=-\infty}^{\infty} h(i)y(n-i))}{\partial h(j)} \right]$$

$$= E \left[ 2e(n) \frac{\partial (-h(j)y(n-j))}{\partial h(j)} \right]$$

$$= -2E [e(n)y(n-j)]$$
Optimal solution: Normal equations

- Consider the MSE $\xi(h) = E \left[ (e(n))^2 \right]$
- The optimal filter satisfies $\nabla \xi(h) |_{h_{\text{opt}}} = 0$. Equivalently, for all $j = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\frac{\partial \xi}{\partial h(j)} = E \left[ 2e(n) \frac{\partial e(n)}{\partial h(j)} \right]$$

$$= E \left[ 2e(n) \frac{\partial (x(n) - \sum_{i=-\infty}^{\infty} h(i)y(n-i))}{\partial h(j)} \right]$$

$$= E \left[ 2e(n) \frac{\partial (-h(j)y(n-j))}{\partial h(j)} \right]$$

$$= -2E [e(n)y(n-j)]$$

- Hence, the optimal filter solves the “normal equations”

$$E [e(n)y(n-j)] = 0, j = \ldots, -2, -1, 0, 1, 2, \ldots$$
Optimal solution: Wiener-Hopf equations

The error of $h_{opt}$ is orthogonal to its observations, i.e., for all $j \in \mathbb{Z}$

$$E[e_{opt}(n)y(n-j)] = 0$$

which is known as “the principle of orthogonality”.
Optimal solution: Wiener-Hopf equations

- The error of $h_{\text{opt}}$ is orthogonal to its observations, i.e., for all $j \in \mathbb{Z}$

$$E [e_{\text{opt}}(n)y(n - j)] = 0$$

which is known as “the principle of orthogonality”.

- Furthermore,

$$E [e_{\text{opt}}(n)y(n - j)] = E \left[ \left( x(n) - \sum_{i=-\infty}^{\infty} h_{\text{opt}}(i)y(n - i) \right) y(n - j) \right]$$

$$= E [x(n)y(n - j)] - \sum_{i=-\infty}^{\infty} h_{\text{opt}}(i)E [y(n - i)y(n - j)] = 0$$
Optimal solution: Wiener-Hopf equations

- The error of $h_{opt}$ is orthogonal to its observations, i.e., for all $j \in \mathbb{Z}$

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$$= E [x(n)y(n - j)] - \sum_{i=-\infty}^{\infty} h_{opt}(i)E [y(n - i)y(n - j)] = 0$$

Result (Wiener-Hopf equations)

$$\sum_{i=-\infty}^{\infty} h_{opt}(i)r_{yy}(i - j) = r_{xy}(j)$$
The Wiener filter

- Wiener-Hopf equations can be solved indirectly, in the complex spectral domain:

\[ h_{opt}(n) \ast r_{yy}(n) = r_{xy}(n) \leftrightarrow H_{opt}(z)P_{yy}(z) = P_{xy}(z) \]
The Wiener filter

- Wiener-Hopf equations can be solved indirectly, in the complex spectral domain:

\[ h_{opt}(n) \ast r_{yy}(n) = r_{xy}(n) \leftrightarrow H_{opt}(z)P_{yy}(z) = P_{xy}(z) \]

Result (The Wiener filter)

\[ H_{opt}(z) = \frac{P_{xy}(z)}{P_{yy}(z)} \]
The Wiener filter

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\[ h_{opt}(n) \ast r_{yy}(n) = r_{xy}(n) \leftrightarrow H_{opt}(z)P_{yy}(z) = P_{xy}(z) \]

**Result (The Wiener filter)**

\[ H_{opt}(z) = \frac{P_{xy}(z)}{P_{yy}(z)} \]

- The optimal filter has an infinite impulse response (IIR), and, is non-causal, in general.
Causal Wiener filter

- We project the unconstrained solution $H_{opt}(z)$ onto the set of causal and stable IIR filters by a two-step procedure:
- First, factorize $P_{yy}(z)$ into causal (right sided) $Q_{yy}(z)$, and anti-causal (left sided) parts $Q_{yy}^*(1/z^*)$, i.e.,
  $$P_{yy}(z) = \sigma_y^2 Q_{yy}(z) Q_{yy}^*(1/z^*).$$
- Select the causal (right sided) part of $P_{xy}(z)/Q_{yy}^*(1/z^*)$.

**Result (Causal Wiener filter)**

$$H_{opt}^+(z) = \frac{1}{\sigma_y^2 Q_{yy}(z)} \left[ \frac{P_{xy}(z)}{Q_{yy}^*(1/z^*)} \right]_+$$
FIR Wiener-Hopf equations

Figure 5: A finite impulse response (FIR) estimator.

- Wiener-Hopf equations for the FIR optimal filter of $N$ taps:

\[
\sum_{i=0}^{N-1} h_{opt}(i)r_{yy}(i - j) = r_{xy}(j), \text{ for } j = 0, 1, \ldots, N - 1.
\]
FIR Wiener Filter


\[
\begin{bmatrix}
  r_{yy}(0) & r_{yy}(1) & \ldots & r_{yy}(N-1) \\
  r_{yy}(1) & r_{yy}(0) & \ldots & r_{yy}(N-2) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{yy}(N-1) & r_{yy}(N-2) & \ldots & y(0)
\end{bmatrix}
\begin{bmatrix}
  h(0) \\
  h(1) \\
  \vdots \\
  h(N-1)
\end{bmatrix}
= \begin{bmatrix}
  r_{xy}(0) \\
  r_{xy}(1) \\
  \vdots \\
  r_{xy}(N-1)
\end{bmatrix}
\]

$\triangleq R_{yy}$: Autocorrelation matrix of $y(n)$ which is Toeplitz.

$\triangleq h_{opt}$

$\triangleq r_{xy}$
FIR Wiener Filter


\[
\begin{bmatrix}
    r_{yy}(0) & r_{yy}(1) & \cdots & r_{yy}(N-1) \\
    r_{yy}(1) & r_{yy}(0) & \cdots & r_{yy}(N-2) \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{yy}(N-1) & r_{yy}(N-2) & \cdots & y(0)
\end{bmatrix}
\begin{bmatrix}
    h(0) \\
    h(1) \\
    \vdots \\
    h(N-1)
\end{bmatrix}
= \begin{bmatrix}
    r_{xy}(0) \\
    r_{xy}(1) \\
    \vdots \\
    r_{xy}(N-1)
\end{bmatrix}
\]

\[\Delta R_{yy} \text{: Autocorrelation matrix of } y(n) \text{ which is Toeplitz.}\]

Result (FIR Wiener filter)

\[h_{opt} = R_{yy}^{-1} r_{xy}.\]
MSE surface

- MSE is a quadratic function of $h$

\[
\xi(h) = h^T R_{yy} h - 2h^T r_{xy} + E \left[ (x(n))^2 \right]
\]

\[
\nabla \xi(h) = 2R_{yy} h - 2r_{xy}
\]

Figure 6: For a 2-tap Wiener filtering example: (a) the MSE surface, (b) gradient vectors.
Example: Wiener equaliser

Figure 7: (a) The Wiener equaliser. (b) Alternative formulation.
Wiener equaliser

Figure 8: Channel equalisation scenario.

- For notational convenience define:

  \[ x'(n) = x(n - d) \]
  \[ e'(n) = x(n - d) - \hat{x}(n - d) \]  \hspace{1cm} (1)

- Label the output of the channel filter as \( y'(n) \) where

  \[ y(n) = y'(n) + \eta(n) \]
Wiener equaliser

Figure 8: Channel equalisation scenario.

- Wiener filter

\[
\mathbf{h}_{opt} = \mathbf{R}_{yy}^{-1} \mathbf{r}_{x'y}
\]  (2)
Wiener equaliser

Figure 8: Channel equalisation scenario.

- **Wiener filter**

\[
\mathbf{h}_{opt} = \mathbf{R}_{yy}^{-1} \mathbf{r}_{x'y}
\]

- The \((i, j)\)th entry in \(\mathbf{R}_{yy}\) is

\[
\mathbf{r}_{yy}(j - i) = E[y(j)y(i)] = E[(y'(j) + \eta(j))(y'(i) + \eta(i))]
\]

\[
= r_{y'y}(j - i) + \sigma_\eta^2 \delta(j - i)
\]

\[
\leftrightarrow \mathbf{P}_{yy}(z) = \mathbf{P}_{y'y'}(z) + \sigma_\eta^2
\]
Wiener equaliser

Figure 8: Channel equalisation scenario.

- Remember \( y'(n) = c(n) \ast x(n) \)
  \[
  \leftrightarrow r_{yy'} = c(n) \ast c(-n) \ast r_{xx}(n) \leftrightarrow P_{y'y'}(z) = C(z)C(z^{-1})P_{xx}(z)
  \]
- Consider a white data sequence \( x(n) \), i.e.,
  \[
  r_{xx}(n) = \sigma_x^2 \delta(n) \leftrightarrow P_{xx}(z) = \sigma_x^2.
  \]
- Then, the complex spectra of the autocorrelation sequence of interest is
  \[
P_{yy}(z) = P_{y'y'}(z) + \sigma_x^2 = C(z)C(z^{-1})\sigma_x^2 + \sigma_\eta^2
  \]
Wiener equaliser

Wiener filter

\[
h_{opt} = R_{yy}^{-1} r_{x'y} \quad (3)
\]
Wiener equaliser

Figure 8: Channel equalisation scenario.

- Wiener filter

\[ h_{opt} = R_{yy}^{-1} r_{x'y} \]  \hspace{1cm} (3)

- The \((j)\)th entry in \( r_{x'y} \) is

\[
r_{x'y}(j) = E \left[ x'(n)y(n - j) \right] \\
= E \left[ x(n - d)(y'(n - j) + \eta(n - j)) \right] \\
= r_{xy'}(j - d) \\
\leftrightarrow P_{x'y}(z) = P_{xy'}(z)z^{-d} \]  \hspace{1cm} (4)
**Wiener equaliser**

![Diagram of Wiener equaliser]

**Figure 8: Channel equalisation scenario.**

- Remember $y'(n) = c(n) \ast x(n)$

  $$\leftrightarrow r_{xy'} = c(-n) \ast r_{xx}(n) \leftrightarrow P_{xy'}(z) = C(z^{-1})P_{xx}(z)$$

- Then, the complex spectra of the cross correlation sequence of interest is

  $$P_{x'y}(z) = P_{xy'}(z)z^{-d} = \sigma_x^2 C(z^{-1})z^{-d}$$
### Wiener equaliser

- Suppose that $\mathbf{c} = [c(0) = 0.5, c(1) = 1]^T \leftrightarrow C(z) = (0.5 + z^{-1})$

- Then,

  $P_{yy}(z) = C(z)C(z^{-1})\sigma_x^2 + \sigma_\eta^2 = (0.5 + z^{-1})(0.5 + z)\sigma_x^2 + \sigma_\eta^2$

  $P_{x'y}(z) = \sigma_x^2 C(z^{-1})z^{-d} = (0.5z^{-d} + z^{-d+1})\sigma_x^2$

- Suppose that $d = 1$, $\sigma_x^2 = 1$, and, $\sigma_\eta^2 = 0.1$

  $r_{yy}(0) = 1.35$, $r_{yy}(1) = 0.5$, and $r_{yy}(2) = 0$

  $r_{x'y}(0) = 1$, $r_{x'y}(1) = 0.5$, and $r_{x'y}(2) = 0$

- The Wiener filter is obtained as

  $$h_{opt} = \left( \begin{bmatrix} 1.35 & 0.5 & 0 \\ 0.5 & 1.35 & 0.5 \\ 0 & 0.5 & 1.35 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.13 \\ -0.05 \end{bmatrix}$$

- The MSE is found as $\xi(h_{opt}) = \sigma_x^2 - h_{opt}^T r_{x'y} = 0.24$. 
Adaptive filtering - Introduction

For notational convenience, define
\[ y(n) \triangleq [y(n), y(n-1), \ldots, y(n-N+1)]^T, \quad h(n) \triangleq [h_0, h_1, \ldots, h_{N-1}]^T \]

The output of the adaptive filter is
\[ \hat{x}(n) = h^T(n)y(n) \]

Optimum solution
\[ h_{\text{opt}} = R_{yy}^{-1} r_{xy} \]
Recursive least squares

- Minimise cost function

\[ \xi(n) = \sum_{k=0}^{n} (x(k) - \hat{x}(k))^2 \]  

(5)

- Solution

\[ R_{yy}(n)h(n) = r_{xy}(n) \]

- LS “autocorrelation” matrix

\[ R_{yy}(n) = \sum_{k=0}^{n} y(k)y^T(k) \]

- LS “cross-correlation” vector

\[ r_{xy}(n) = \sum_{k=0}^{n} y(k)x(k) \]
Recursive least squares

- Recursive relationships

\[ R_{yy}(n) = R_{yy}(n - 1) + y(n)y^T(n) \]
\[ r_{xy}(n) = r_{xy}(n - 1) + y(n)x(n) \]

- Substitute for \( r_{xy} \)

\[ R_{yy}(n)h(n) = R_{yy}(n - 1)h(n - 1) + y(n)x(n) \]

- Replace \( R_{yy}(n - 1) \)

\[ R_{yy}(n)h(n) = \left( R_{yy}(n)h(n) - y(n)y^T(n) \right) h(n - 1) + y(n)x(n) \]

- Multiple both sides by \( R_{yy}^{-1}(n) \)

\[ h(n) = h(n - 1) + R_{yy}^{-1}(n)y(n)e(n) \]
\[ e(n) = x(n) - h^T(n - 1)y(n) \]
Recursive least squares

- Recursive relationships

\[ R_{yy}(n) = R_{yy}(n - 1) + y(n)y^T(n) \]

- Apply Sherman-Morrison identity

\[ R_{yy}^{-1}(n) = R_{yy}^{-1}(n - 1) - \frac{R_{yy}^{-1}(n - 1)y(n)y^T(n)R_{yy}^{-1}(n - 1)}{1 + y^T(n)R_{yy}^{-1}(n - 1)y(n)} \]
### Summary

Recursive least squares (RLS) algorithm:

1. $R_{yy}(0) = \frac{1}{\delta} I_N$ with small positive $\delta$ \> Initialisation 1
2. $h(0) = 0$ \> Initialisation 2
3. For $n = 1, 2, 3, \ldots$ do \> Iterations
4. $\hat{x}(n) = h^T(n-1)y(n)$ \> Estimate $x(n)$
5. $e(n) = x(n) - \hat{x}(n)$ \> Find the error
6. $R_{yy}^{-1}(n) = \frac{1}{\alpha} \left( R_{yy}^{-1}(n-1) - \frac{R_{yy}^{-1}(n-1)y(n)y^T(n)R_{yy}^{-1}(n-1)}{\alpha + y^T(n)R_{yy}^{-1}(n-1)y(n)} \right)$ \> Update the inverse of the autocorrelation matrix
7. $h(n) = h(n-1) + R_{yy}^{-1}(n)y(n)e(n)$ \> Update the filter coefficients
8. End for
Stochastic gradient algorithms

- MSE contour - 2-tap example:

Figure 10: Method of steepest descent.
Steepest descent

- MSE contour - 2-tap example:

\[ h_1 \]

\[ h_0 \]

The gradient vector

\[
\nabla_h(n) = \left[ \frac{\partial \xi}{\partial h(0)}, \frac{\partial \xi}{\partial h(1)}, \ldots, \frac{\partial \xi}{\partial h(N - 1)} \right]^T
\]

\[ = 2R_{yy}h(n) - 2r_{xy} \]
Steepest descent

- MSE contour - 2-tap example:

- Update initial guess in the direction of steepest descent:
  \[ h(n + 1) = h(n) - \mu \nabla h(n) \]

- Step-size \( \mu \).
Steepest descent

- MSE contour - 2-tap example:

- Gradient at new guess.
Convergence of steepest descent

- MSE contour - 2-tap example:

\[ h(n + 1) = h(n) - \mu \nabla h(n) \]

\[ \nabla h(n) = \left[ \frac{\partial \xi}{\partial h(0)}, \frac{\partial \xi}{\partial h(1)}, \ldots, \frac{\partial \xi}{\partial h(N - 1)} \right]^T \]

\[ \left. \frac{\partial \xi}{\partial h(n)} \right|_{h = h(n)} = 2R_{yy}h(n) - 2r_{xy} \]

\[ 0 < \mu < \frac{1}{\lambda_{\max}} \]
Stochastic gradient algorithms

- A time recursion:

\[ h(n + 1) = h(n) - \mu \hat{\nabla}_h(n) \]

- The exact gradient:

\[ \nabla_h(n) = -2E \left[ y(n)(x(n) - y(n)^T h(n)) \right] \]
\[ = -2E [y(n)e(n)] \]

- A simple estimate of the gradient

\[ \hat{\nabla}_h(n) = -2y(n + 1)e(n + 1) \]

- The error

\[ e(n + 1) = x(n + 1) - h(n)^T y(n + 1) \]
The Least-mean-squares (LMS) algorithm:

1: $h(0) = 0$  \hspace{1cm} \triangleright \text{Initialisation}
2: \textbf{for} $n = 1, 2, 3, \ldots$ \textbf{do}  \hspace{1cm} \triangleright \text{Iterations}
3: $\hat{x}(n) = h^T(n-1)y(n)$  \hspace{1cm} \triangleright \text{Estimate $x(n)$}
4: $e(n) = x(n) - \hat{x}(n)$  \hspace{1cm} \triangleright \text{Find the error}
5: $h(n) = h(n-1) + 2\mu y(n)e(n)$  \hspace{1cm} \triangleright \text{Update the filter coefficients}$
6: \textbf{end for}
LMS block diagram

Figure 11: Least mean-square adaptive filtering.
Convergence of the LMS

- MSE contour - 2-tap example:

- Eigenvalues of $R_{yy}$ (in this example, $\lambda_0$ and $\lambda_1$).
- The largest time constant $\tau_{max} > \frac{\lambda_{max}}{2\lambda_{min}}$.
- Eigenvalue ratio (EVR) is $\frac{\lambda_{max}}{\lambda_{min}}$.
- Practical range for step-size $0 < \mu < \frac{1}{3N\sigma_y^2}$.
Eigenvalue ratio (EVR)

Figure 12: Eigenvectors, eigenvalues and convergence: (a) the relationship between eigenvectors, eigenvalues and the contours of constant MSE; (b) steepest descent for EVR of 2; (c) EVR of 4.
Comparison of RLS and LMS

Error vector norm

\[ \rho(n) = E \left[ (h(n) - h_{opt})^T (h(n) - h_{opt}) \right] \]
Comparison: Performance

![Convergence plots for N = 16 taps adaptive filtering in the system identification configuration: EVR = 1 (i.e., the impulse response of the noise shaping filter is $\delta(n)$).]

**Figure 14:** Convergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: $EVR = 1$ (i.e., the impulse response of the noise shaping filter is $\delta(n)$).
Comparison: Performance

Figure 15: Convergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: EVR = 11.
Comparison: Performance

Figure 16: Convergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: EVR (and, correspondingly the spectral coloration of the input signal) progressively increases to 68.
Comparison: Complexity

Table: Complexity comparison of $N$-point FIR filter algorithms.

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<th>Computational load</th>
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<td>BLMS (via FFT)</td>
<td>$10\log(N)+8$</td>
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Applications

Adaptive filtering algorithms can be used in all application areas of optimal filtering.

Some examples:
- Adaptive line enhancement
- Adaptive tone suppression
- Echo cancellation
- Channel equalisation
Figure 17: Adaptive filtering configurations: (a) direct system modelling; (b) inverse system modelling; (c) linear prediction.
Adaptive line enhancement

![Diagram of signal spectrum and system configuration]

**Figure 18:** Adaptive line enhancement: (a) signal spectrum; (b) system configuration.
Adaptive predictor

- Prediction filter: $a_0 + a_1 z^{-1} + a_2 z^{-2}$
- Prediction error filter: $1 - a_0 z^{-1} - a_1 z^{-2} - a_2 z^{-3}$
Adaptive tone suppression

Figure 19: Adaptive tone suppression: (a) signal spectrum; (b) system configuration.
Adaptive noise whitening

(a) Input spectrum

(b) System configuration

Figure 20: Adaptive noise whitening: (a) input spectrum; (b) system
Echo cancellation

- A typical telephone connection

- Hybrid transformers to route signal paths.
Echo cancellation (contd)

- Echo paths in a telephone system

![Diagram of telephone system showing near and far echo paths](image)

- Near and far echo paths.
Echo cancellation (contd)

- Fixed filter?
Echo cancellation (contd)

Figure 21: Application of adaptive echo cancellation in a telephone handset.
Channel equalisation

Figure 22: Adaptive equaliser system configuration.

- Simple channel
  \[ y(n) = \pm h_0 + \text{noise} \]

- Decision circuit
  \[
  \text{if } y(n) \geq 0 \text{ then } x(n) = +1 \text{ else } x(n) = -1
  \]

- Channel with intersymbol interference (ISI)
  \[
  y(n) = \sum_{i=0}^{2} h_i x(n - i) + \text{noise}
  \]
Channel equalisation

![Decision directed equaliser](image)

**Figure 23**: Decision directed equaliser.
Optimal signal detection

- Signal detection as 2-ary (binary) hypothesis testing:

\[ H_0 : y(n) = \eta(n) \]
\[ H_1 : y(n) = x(n) + \eta(n) \]  \hspace{1cm} (8)

- In a sense, decide which of the two possible ensembles \( y(n) \) is generated from.
- Finite length signals, i.e.,

\[ n = 0, 1, 2, \ldots, N - 1 \]

- Vector notation

\[ H_0 : \mathbf{y} = \eta \]
\[ H_1 : \mathbf{y} = \mathbf{x} + \eta \]
Bayesian hypothesis testing

- Having observed \( y = \bar{y} \), find the probabilities of \( H_1 \) and \( H_0 \) and decide on the hypothesis with the maximum value.

- Equivalently,
  i) consider a random variable \( H \in \{H_0, H_1\} \) and find its posteriori distribution:

\[
P(H = H_i | \bar{y}) = \frac{p(\bar{y} | H = H_i)p(H = H_i)}{p(\bar{y} | H = H_0)p(H = H_0) + p(\bar{y} | H = H_1)p(H = H_1)}
\]

for \( i = 0, 1 \).

ii) Find the maximum a-posteriori (MAP) estimate of \( H \).

\[
\hat{H} = \arg \max_H p(H | \bar{y})
\]
Detection of deterministic signals - white Gaussian noise

- $\mathbf{x}$ is a known vector, $\eta \sim \mathcal{N}(\cdot; \mathbf{0}, \sigma^2 \mathbf{I})$.
- MAP decision as a likelihood ratio test:

$$
\begin{align*}
H_1 & \quad p(H_1 | \tilde{y}) > p(H_0 | \tilde{y}) \\
H_0 & \quad p(H_0 | \tilde{y}) > p(H_1 | \tilde{y})
\end{align*}
$$

$$
\begin{align*}
H_1 & \quad p(\tilde{y} | H_1) P(H_1) > p(\tilde{y} | H_0) P(H_0) \\
H_0 & \quad p(\tilde{y} | H_0) P(H_0) > p(\tilde{y} | H_1) P(H_1)
\end{align*}
$$

$$
\begin{align*}
H_1 & \quad \frac{p(\tilde{y} | H_1)}{p(\tilde{y} | H_0)} P(H_0) > P(H_1) \\
H_0 & \quad \frac{p(\tilde{y} | H_0)}{p(\tilde{y} | H_1)} P(H_1) > P(H_0)
\end{align*}
$$

$$
\begin{align*}
\mathcal{N}(\tilde{y} - \mathbf{x}; \mathbf{0}, \sigma^2 \mathbf{I}) & > \mathcal{N}(\tilde{y}; \mathbf{0}, \sigma^2 \mathbf{I}) \\
H_1 & \quad P(H_0) > P(H_1) \\
H_0 & \quad P(H_1) > P(H_0)
\end{align*}
$$
Detection of deterministic signals - AWGN (contd)

- The numerator and denominator of the likelihood ratio are

$$p(\bar{y}|H_1) = \mathcal{N}(\bar{y} - x; 0, \sigma^2 I)$$

$$= \frac{1}{(2\pi\sigma^2)^N/2} \prod_{n=0}^{N-1} \exp \left\{ -\frac{(\bar{y}(n) - x(n))^2}{2\sigma^2} \right\}$$

$$= \frac{1}{(2\pi\sigma^2)^N/2} \exp \left\{ -\frac{1}{2\sigma^2} \left( \sum_{n=0}^{N-1} (\bar{y}(n) - x(n))^2 \right) \right\}$$

(9)

- Similarly

$$p(\bar{y}|H_0) = \mathcal{N}(\bar{y}; 0, \sigma^2 I)$$

$$= \frac{1}{(2\pi\sigma^2)^N/2} \exp \left\{ -\frac{1}{2\sigma^2} \left( \sum_{n=0}^{N-1} (\bar{y}(n))^2 \right) \right\}$$

(10)

- Therefore

$$\frac{p(\bar{y}|H_1)}{p(\bar{y}|H_0)} = \exp \left\{ \frac{1}{\sigma^2} \left( \sum_{n=0}^{N-1} (\bar{y}(n)x(n) - \frac{1}{2}x(n)^2) \right) \right\}$$

(11)
Detection of deterministic signals - AWGN (contd)

- Take the logarithm of both sides of the likelihood ratio test

\[
\log \exp \left\{ \frac{1}{\sigma^2} \left( \sum_{n=0}^{N-1} (\bar{y}(n)x(n) - \frac{1}{2}x(n)^2) \right) \right\} \begin{cases} H_1 & \text{if} \ \frac{H_1}{H_0} \gg 1 \\ H_0 & \text{if} \ \frac{H_1}{H_0} \ll 1 \\ \end{cases} \log \frac{P(H_0)}{P(H_1)}
\]

- Now, we have a linear statistical test

\[
\sum_{n=0}^{N-1} \bar{y}(n)x(n) \begin{cases} H_1 & \text{if} \ \frac{H_1}{H_0} \gg 1 \\ H_0 & \text{if} \ \frac{H_1}{H_0} \ll 1 \\ \end{cases} \sigma^2 \log \frac{P(H_0)}{P(H_1)} + \frac{1}{2} \sum_{n=0}^{N-1} x(n)^2
\]

\[\triangleq \tau: \text{Decision threshold}\]

\[
y(n) \rightarrow h_{MF}(n) = x(N - 1 - n) \]

\[
\begin{cases} H_1 & \text{if} \ n = N - 1 \\ H_0 & \text{if} \ n < \tau \end{cases}\]

\[\text{Decision variable} \ \\ \hat{H}
\]
Detection of deterministic signals under coloured noise

- For the case, \( \eta \sim \mathcal{N}(\cdot; 0, C_\eta) \).

The whitening filter can be designed in the optimal filtering framework or an adaptive algorithm can be used.

The results on optimal detection under white Gaussian noise holds for the signal \( x(n) \ast h_\eta(n) \).
Summary

- Optimal filtering: Problem statement
- General solution via Wiener-Hopf equations
- FIR Wiener filter
- Adaptive filtering as an online optimal filtering strategy
- Recursive least-squares (RLS) algorithm
- Least mean-square (LMS) algorithm
- Application examples
- Optimal signal detection via matched filtering
Further reading