Optimality Criteria for Adaptive Waveform Design in MIMO Radar Systems

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***Abstract*—Adaptive waveform design in MIMO radar can improve the estimation of target parameters. Comparing the relative benefits of known optimality criteria, namely the A-optimal, D-optimal and E-optimal criteria, is important to inform the best implementation of actual MIMO radar adaptive waveform design systems. In this paper we provide such a comparison, both theoretically and by assessing simulated numerical results. Whilst all three criteria are complementary, and thus all potentially could have a role in adaptive waveform design for MIMO radar, our results indicate that a system which generally uses A-optimal design, but switches to E-optimal design in some circumstances may be suitable for practical application.**

I. Introduction

Adaptive waveform design for active sensing leverages information acquired from the received reflected signals to inform the design of future waveform transmissions to maximise, in some sense, the information we can expect to obtain. One particularly fertile field for application of such techniques is MIMO radar, and the types of techniques we consider are sometimes included under the broad term *cognitive radar* [1]–[3]. An important quantity for such design is the expected covariance matrix, Γ_k, given a particular waveform design at the 'k^{th}' step, which can be expressed (from [4, Eq. (11)]):

$$\Gamma_k = \iint (\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)^T \\ p(\boldsymbol{\theta}_k|\mathbf{X}^{k-1}, \mathbf{S}^{k-1}) p(\mathbf{X}_k|\boldsymbol{\theta}_k, \mathbf{S}_k)\, \mathrm{d}\mathbf{X}_k\ \mathrm{d}\boldsymbol{\theta}_k, \quad (1)$$

where $\hat{\boldsymbol{\theta}}_k = \mathbb{E}(\boldsymbol{\theta}_k|\mathbf{X}_k, \mathbf{X}^{k-1}, \mathbf{S}_k, \mathbf{S}^{k-1})$, i.e., the expected estimate of $\boldsymbol{\theta_k}$, itself a random variable. The other terms are related by the standard MIMO radar formulation (with N_T transmission elements, N_R receiving elements and L snapshots per step:

$$\mathbf{X}_k = \mathbf{H}_k(\boldsymbol{\theta}_k)\mathbf{S}_k + \mathbf{N}_k, \quad (2)$$

where $\mathbf{S}_k \in \mathbb{C}^{N_T \times L}$ is the transmitted waveform, $\mathbf{X}_k \in \mathbb{C}^{N_R \times L}$ is the received waveform and $\mathbf{H}_k(\boldsymbol{\theta}_k) \in \mathbb{C}^{N_R \times N_T}$ represents the channel response as a non-linear function (in general) of $\boldsymbol{\theta}_k$, a vector of the Q parameters of the target, i.e., $\boldsymbol{\theta}_k \in \mathbb{C}^{Q \times 1}$, and $\mathbf{N}_k \in \mathbb{C}^{N_R \times L}$ represents additive white Gaussian noise. See [4], [5] for full details of this formulation of MIMO radar.

An important goal in adaptive waveform design is minimisation of the variance of the next parameter estimation, formally:

$$\begin{array}{lll} \text{minimise:} & \Gamma_k & \text{w.r.t.}\mathbf{S}_k \\ \text{subject to:} & \mathrm{tr}\left(\frac{1}{L}\mathbf{S}_k\mathbf{S}_k^H\right) \leq \mathrm{P} & \end{array} \quad (3)$$

where $\mathrm{tr}(.)$ is the matrix trace operation and P is the total transmit power per step, i.e., the constraint is a maximum power constraint. As Γ_k is a matrix (i.e., the expected covariance matrix), this is not a strictly meaningful expression, and it is widely accepted that there are a number of relevant metrics associated with the expected covariance matrix which can be minimised, namely its trace, determinant or largest eigenvalue [4], [5]. It should be noted that the first two of these three have been addressed for the linear-Gaussian case [6].

In this paper, we apply all three of these criteria to a MIMO radar set-up based on that in [4] (i.e., not linear-Gaussian) and discuss the distinction between them in terms of practical MIMO radar considerations. This enables us to assess the suitability of the three criteria for MIMO radar applications, and thus recommend an appropriate method for optimal waveform design in MIMO radar systems.

A significant technical obstacle that we had to overcome was that, whilst the cost-function to optimise has been analytically expressed (2), in general its derivative cannot be efficiently computed (except for trace optimisation, as in [4]). Therefore it was necessary to use an optimisation technique that did not require evaluation of the gradient of the cost-function and this was achieved by using a random search over the waveform design space.

In general, however, we recognise that random search may not be a computationally efficient way to optimise the waveform design, thus a further contribution of this work is that it provides motivation to investigate how best to optimise adaptive waveform designs in MIMO radar systems. That is, in light of the fact that the results and discussion herein indicate that D and E optimal adaptive waveform designs (which cannot be computed using gradient methods) *do* have a place in actual MIMO radar systems. Moreover, even for A-optimal design, we note that general search methods (i.e., including meta-heuristic optimisation, grid search and random search) have a larger search area than gradient methods, and thus may better optimise the cost function which is multi-modal. It is also acknowledged that the heavy computational complexity associated with the calculation of the cost-function gradient is one of the main potential barriers to the feasibility of adopting such adaptive waveform design methods in actual MIMO radar systems [4].

II. Overview of Optimality Criteria

The field of adaptive waveform design is closely related to that of optimal experimental design [7], and it adds some value to interpret our three optimisation criteria in terms of the latter. In particular, minimisation of the trace of the expected covariance matrix is denoted 'A-optimality', and corresponds to minimisation of the expected mean squared error. For this reason it is also referred to 'estimation-theoretic optimisation'. Within the field of adaptive waveform design, A-optimal design is typically the optimisation metric of choice, owing to its straightforward interpretation and the elegance of its analytic expression (i.e., [4, Eq. (11)]) [4], [5].

Minimisation of the determinant of the expected covariance matrix is termed 'D-optimal design' in optimal experimental design nomenclature, and it can be thought of in terms of information theoretic concerns. Specifically, in the linear-Gaussian case minimising the determinant maximises the mutual information (MI) between the target parameters and the received signal [6]. Maximisation of MI is in general, an important metric for adaptive waveform design, but not one we explicitly consider here as it has already been well addressed [6], [8]–[10].

Finally, minimisation of the largest eigenvector is denoted 'E-optimality' in optimal experimental design nomenclature, and is the least researched in the context of application to adaptive waveform design. E-optimality designs the waveform to minimise the worst case variance, and in MIMO radar this may be dominated by parameters associated with one particle target (for example a more distant target whose reflected signal has a lower amplitude), and accordingly we would expect the waveform to be designed to better resolve the parameters of this target. So it follows that the theoretical notion of E-optimality may have interesting practical application to adaptive waveform design in MIMO radar.

Note that optimal experimental design literature generally considers the inverse of the expected variance matrix, known as the 'information matrix', and thus A-optimality corresponds to trace maximisation of the information matrix, D-optimality corresponds to maximisation of the determinant of the information matrix and E-optimality corresponds to maximisation of the minimum eigenvalue of the information matrix. The definitions we use herein are equivalent to these, and more relevant for our application.

III. Implementation of a General Optimisation Framework for General Optimal Design Metric

As in [4], we consider adaptive waveform design in a Bayesian setting where the posterior probability density function (PDF) of $\boldsymbol{\theta}_k$ is approximated by a finite size probability mass function:

$$p(\boldsymbol{\theta}_k|\mathbf{X}^{k-1},\mathbf{S}^{k-1}) \approx \sum_{i=1}^{N_P} w_k^{(i)} \delta(\boldsymbol{\theta}_k - \boldsymbol{\theta}_k^{(i)}), \tag{4}$$

for some N_P, where $0 \leq w_k^{(i)} \leq 1$ is the weight of the i^{th} particle. Additionally, a further pair of random variables is defined:

$$\begin{array}{lcl} \boldsymbol{\theta}_k'^{(m)} & \sim & \sum_{i=1}^{N_P} w_k^{(i)} \delta(\boldsymbol{\theta}_k'^{(m)} - \boldsymbol{\theta}_k^{(i)}) \\ \mathbf{X}_k^{(m)} & \sim & p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k'^{(m)}, \mathbf{S}_k(0)) \end{array}, \tag{5}$$

for the m^{th} sample, where $\mathbf{S}_k(0)$ is a fixed realisation of $\mathbf{S}_k$. This enables (1) to be numerically approximated:

$$\begin{aligned}\Gamma_k \approx \sum_{m=1}^{N_S} & \left(\frac{\sum_{i=1}^{N_P} w_k^{(i)} p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k^{(i)}, \mathbf{S}_k)\boldsymbol{\theta}_k^{(i)}}{\sum_{i=1}^{N_P} w_k^{(i)} p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k^{(i)}, \mathbf{S}_k)} - \boldsymbol{\theta}_k'^{(m)} \right) \times \\ & \left(\frac{\sum_{i=1}^{N_P} w_k^{(i)} p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k^{(i)}, \mathbf{S}_k)\boldsymbol{\theta}_k^{(i)}}{\sum_{i=1}^{N_P} w_k^{(i)} p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k^{(i)}, \mathbf{S}_k)} - \boldsymbol{\theta}_k'^{(m)} \right)^T \times \\ & \frac{p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k'^{(m)}, \mathbf{S}_k)/p(\mathbf{X}_k^{(m)}|\boldsymbol{\theta}_k'^{(m)}, \mathbf{S}_k(0))}{\sum_{m'=1}^{N_S} p(\mathbf{X}_k^{(m')}|\boldsymbol{\theta}_k'^{(m')}, \mathbf{S}_k)/p(\mathbf{X}_k^{(m')}|\boldsymbol{\theta}_k'^{(m')}, \mathbf{S}_k(0))}, \end{aligned} \tag{6}$$

where N_S samples of the pair $(\boldsymbol{\theta}_k'^{(m)}, \mathbf{X}_k^{(m)})$ are drawn from (5) (see [4] for full justification of this approximation). Notice that Γ_k is expressed as the sum of N_S matrices, so it follows that, unlike for the case of trace optimisation (where the cost function can be expressed as a sum of scalars each of which is expressed as a function of $\mathbf{S}_k$), explicit expression of the determinant and largest eigenvalue of (6) in terms of $\mathbf{S}_k$ (i.e., for differentiation) is not trivial, and more importantly would not lead to a computationally feasible expression for the derivative. For this reason, we make the practical decision to optimise the cost function without evaluating the gradient. In addition to random or grid search, there exist myriad 'meta-heuristic' optimisation methods to achieve this [11].

A general definition of the class of algorithms that only require evaluation of the cost function (by definition a maximum of N_C times), and not its derivative, is given in Algorithm 1, in which we denote the type of optimality (A, D or E) as $\mathcal{M}(\Gamma)$

Algorithm 1 General Optimisation Framework

For: $n = 1 : N_C$
  $\mathbf{S}(n) = f(\mathbf{S}(1) \ldots \mathbf{S}(n-1), \mathcal{M}(\Gamma(1)) \ldots \mathcal{M}(\Gamma(n-1)))$
  Evaluate: $\Gamma(n) = \Gamma(\mathbf{S}(n))$ according to (6)
  Evaluate: $\mathcal{M}(\Gamma(n))$
End For
Return: $\Gamma_k = \arg\min(\mathcal{M}(\Gamma(1)) \ldots \mathcal{M}(\Gamma(n-1)))$

From Algorithm 1, we can see that 'f' is the function which tells us how to choose the next $\mathbf{S}_k$ at which to evaluate the cost function. In grid search and random search, this

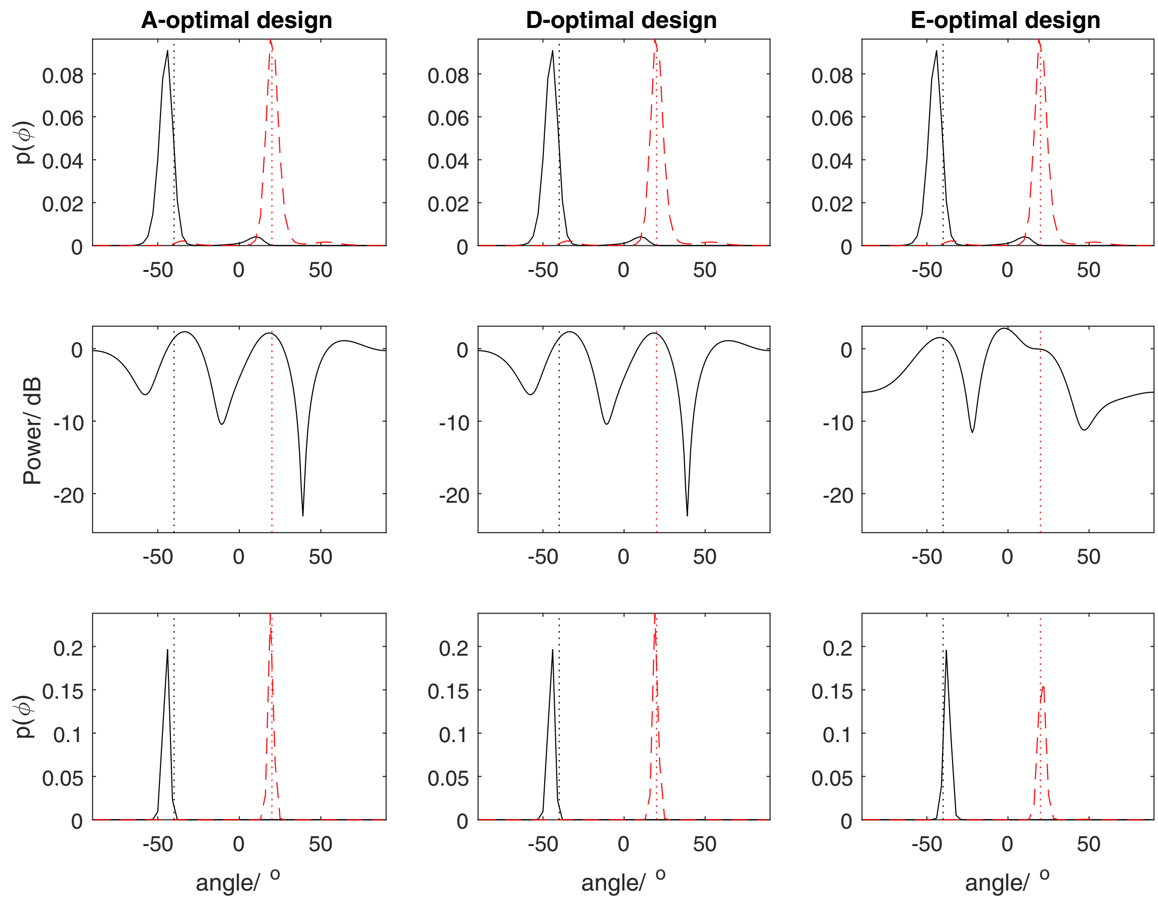

Fig. 1. Example of optimal waveform design for targets with approximately equal prior PDF.

is independent of the previous cost function evaluations, however in meta-heuristic algorithms this information is used to inform the next location at which to evaluate the cost function. Thus Algorithm 1 represents a general optimisation framework for adaptive waveform design, where any combination of meta-heuristic optimisation method and optimisation metric can be chosen.

IV. Numerical Simulations and Results

For the optimisation method, ($f(.)$ in Algorithm 1), we use random search, which suffices to prove the principle that the three optimality criteria lead to different adaptive waveform designs. Additionally, computation was not especially constrained for the numerical simulations provided herein and so a relatively large number (100) of random trial waveform designs could be evaluated for each scenario, and thus it was preferable to do so rather than risk prejudicing the results with a choice of meta-heuristic optimisation method that turned out to be especially suited to only a subset of the three optimality criteria. Moreover, by randomly pre-selecting the ensemble of possible waveforms, and keeping this the same for the three optimality criteria, the optimisation processes for the three criteria were all provided exactly the same candidate waveform, which was useful for drawing conclusions on their differences.

To illustrate the adaptive waveform designs produced by the three optimality criteria, it is preferable to use a multi-target scenario, and to this end we use a two target experimental set-up based on that in [4]. That is, the target parameter vector consists of the two stationary target parameter angles, $\theta = [\phi_1; \phi_2]$, $90^o \leq \phi_1, \phi_2 < 90^o$ (thus $Q = 2$). It follows that, for MIMO radar, (2) can be expressed:

$$\mathbf{X}_k = \sum_{q=1}^{Q} \alpha_q \mathbf{a}_R(\phi_q)\mathbf{a}_T^T(\phi_q)\mathbf{S}_k + \mathbf{N}_k, \qquad (7)$$

where $\mathbf{H}_k = \sum_{q=1}^{Q} \alpha_q \mathbf{a}_R(\phi_q)\mathbf{a}_T^T(\phi_q)$, in which α_q is the complex attenuation of the q^{th} target (treated as known for simplicity) and $\mathbf{a}_R \in \mathbb{C}^{N_R \times 1}$ and $\mathbf{a}_T \in \mathbb{C}^{N_T \times 1}$ are the steering vectors associated with the receive and transmit arrays respectively.

As in [4] we use $N_T = N_R = 5$, $L = 1$, $N_P = 1770$ (a resolution of 3^o), $N_S = 250$ and $[\phi_1; \phi_2] = [-40^o; 20^o]$.

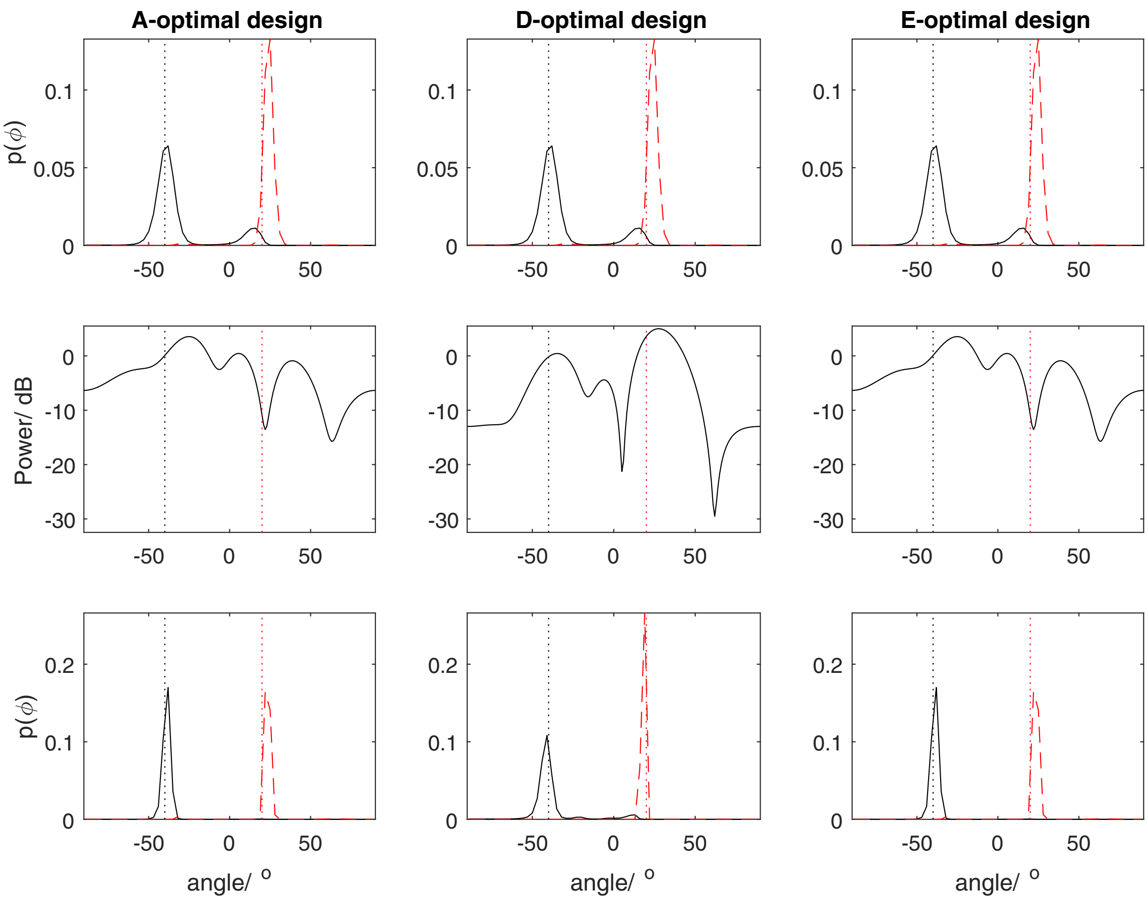

Fig. 2. Example of optimal waveform design for targets with unequal prior PDF.

In general, the system may run for many steps, however for illustrative purposes we show just two steps (i.e., a single waveform design), to achieve this we increase the Array Signal to Noise Ratio (ASNR) such that the first adaptive waveform design is meaningful. We found ASNR $= 6\ dB$ to be a suitable value, where ASNR $\triangleq (|\alpha_1|^2+|\alpha_2|^2)\mathrm{P}N_R L/(0.5\sigma_n^2)$ (in which the factor 0.5 in the denominator is introduced owing to our definition of σ_n^2 as the noise variance for each of the real and imaginary components). We consider two scenarios, firstly 'equal attenuation', where $|\alpha_1| = |\alpha_2|$ and secondly 'unequal attenuation'. where $|\alpha_1| = 0.5|\alpha_2|$. These are shown in Fig. 1 and Fig. 2 respectively. In these plots the marginal PDF of the target location for the target located at -40^o is shown in black, and the marginal PDF of the target location for the target located at 20^o is shown in red.

For the first step, the transmitted waveform is uniformly spread with angle as there is no information to adaptively design the waveform ('orthogonal' transmission), and so it is unnecessary to include this in the plots of the results. Thus the results appear as the PDF after the reflections received from the orthogonal transmission at step one, followed by the waveform design, followed by the PDF after the reflections received from the designed transmission at step two. So it follows that the two PDFs can readily be interpreted as a prior and a posterior respectively. To ensure a fair comparison, we fixed the prior to be the same for all three optimal design criteria in each scenario.

V. Discussion

Considering first the adaptive waveform designs for the case where the two target locations are approximately equally known *a priori*, as exemplified in Fig. 1, we observe that the design processes according to the A-optimal and D-optimal criteria have chosen the same waveform design, whereas that of the E-optimal criteria is different. In the case of the latter, we can see that a greater amount of power has been steered towards the non-zero region of probability density for the target located -40^o around 0^o to 20^o.

Whilst such a conclusion may appear somewhat tenuous in this case, the second example shown in Fig. 2, in which the prior PDF of the target located at -40^o has significantly higher variance than that of the target located at 20^o, reinforces this effect. In this plot, we can see that the design processes according to the A-optimal and E-optimal criteria have chosen the same waveform design, whereas that of the D-optimal criteria is different. Moreover, we can see that, again there is a region of non-zero probability density around 20^o for the target located at -40^o and the A-optimal and E-optimal designs steer a peak to this angle, whereas the D-optimal design steers a null. Additionally, for the D-optimal design the majority of the power is steered in a wide beam towards the (relatively certain) target at 20^o, whereas the A-optimal and E-optimal designs actually steer a null to this angle.

Taking a more general view, it is perhaps ill-advised to rely too heavily on our interpretation of these two examples alone, to draw general conclusions regarding the differences between the three optimality criteria for adaptive waveform design in MIMO radar. Rather, it is merely possible to state, definitively, that the three criteria *have*, definitively, selected different waveforms given the same prior PDF and set of waveforms from which to chose. Thus we are motivated to attempt to interpret this fact in terms of the definitions of the three criteria.

The key to doing this is to realise that the target locations are independent, and thus the expected covariance matrix should be diagonal (or approximately diagonal for the numerical approximation). In this case, the three criteria can be simplified:

$$\text{A-optimal:} \qquad \min\left(\sum_{i=1}^{Q}\sigma_i^2\right),$$

$$\text{D-optimal:} \qquad \min\left(\prod_{i=1}^{Q}\sigma_i^2\right),$$

$$\text{E-optimal:} \qquad \min\left(\max_{i=1:Q}\left(\sigma_i^2\right)\right),$$

where σ_i^2 is the variance of the estimate of the location of the ith target.

This clearly shows the difference between E-optimal design and the other two, as the former is concerned with the variance of the least certain target. We can see evidence of this in the third row of Fig. 2, where the marginal posterior PDFs of the two targets have similar appearance for the E-optimal design, whereas those of the D-optimal design are still very unequal (note that we used the same random noise realisation for consistency).

Such a property potentially has practical implications for real-world MIMO radar applications. For example, one can envisage a system in which the A-optimal or D-optimal criterion was used in general, with the option of switching to use the E-optimal criterion if is desired to improve the parameter estimates associated with the most uncertain target.

The relative benefits of A-optimal and D-optimal design is somewhat more subtle, and still the subject debate in the field of optimal design, however it is worth noting that A-optimal design is computationally more simple (as (6) reduces to a sum of scalars rather than matrices, as in [4]). Furthermore, the results in Fig. 2 suggest that A-optimal design shares the desirable property of E-optimal design of focussing the power on the less certain target, whereas D-optimal design does not necessarily do so. We do, however, realise that this may not be the case for a larger number of targets.

In summary, we tentatively recommend a system in which A-optimal design is generally used, but E-optimal design is switched to if better estimation is required for a particular, relatively uncertain, target.

VI. Conclusions

In this paper we have applied three well known optimal design criteria, namely A-optimal, D-optimal and E-optimal, to the problem of adaptive waveform design in MIMO radar systems. Our results show that the waveforms designed according to the three criteria differ, and moreover that this difference corresponds to what one would expect from physical reasoning. In light of these results we recommend an embodiment for actual MIMO radar systems combining A-optimal and E-optimal design.

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